(2.2)

(2) $C_G(Z(G)) = \{g \in G \mid gh = hg \text{ for all } h \in Z(G)\}$

Since $gh = hg$ for every $h \in Z(G)$ by the def. of $Z(G)$, we have that $C_G(Z(G)) = G$.

Let $g \in G$. Then $gZ(G)g^{-1} = \{gzg^{-1} \mid z \in Z(G)\}$

$= \{gg^{-1}z \mid z \in Z(G)\} = \{z \mid z \in Z(G)\} = Z(G)$.

Thus, $N_G(Z(G)) = \{g \in G \mid gZ(G)g^{-1} = Z(G)\} = G$

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(5) (a) $G = S_3 = \{1, (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$
$A = \{1, (1,2,3), (1,3,2)\}$

$\left.\begin{array}{l}(1,2)(1,2,3) = (2,3) \\ (1,2,3)(1,2) = (1,3)\end{array}\right\}$ So, $(1,2) \notin C_{S_3}(A)$

$\left.\begin{array}{l}(2,3)(1,2,3) = (1,3) \\ (1,2,3)(2,3) = (1,2)\end{array}\right\}$ So, $(2,3) \notin C_{S_3}(A)$

$\left.\begin{array}{l}(1,3)(1,2,3) = (1,2) \\ (1,2,3)(1,3) = (2,3)\end{array}\right\}$ So, $(1,3) \notin C_{S_3}(A)$

Since A is an abelian group ($A = \langle (1,2,3) \rangle$ is cyclic) $ab = ba$ for every $a, b \in A$. Hence $A \subseteq C_{S_3}(A)$.
Thus, $A = C_{S_3}(A)$.

5(a) continued...

$$(1,2)A(1,2)^{-1} = (1,2)A(1,2)$$
$$= \{(1,2)1(1,2),\ (1,2)(1,2,3)(1,2),\ (1,2)(1,3,2)(1,2)\}$$
$$= \{1,\ (1,3,2),\ (1,2,3)\} = A$$

Similarly, $(1,3)A(1,3)^{-1} = A$, $(2,3)A(2,3)^{-1} = A$, $1A1^{-1} = A$, $(1,2,3)A(1,2,3)^{-1} = A$, and $(1,3,2)A(1,3,2)^{-1} = A$.

So, $N_G(A) = G$.

6(a) Let H be a subgroup of G. We show that $H \leq N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

proof: We just need to show that H is a subset of $N_G(H)$ since we are already assuming that H is a subgroup.

$\boxed{\subseteq}$ Let $h \in H$. Then, $hHh^{-1} = \{hh_1h^{-1} \mid h_1 \in H\}$. Since H is a subgroup, if $h, h_1 \in H$, then $hh_1h^{-1} \in H$, because H is closed under the group operation and inversion. Hence, $hHh^{-1} \subseteq H$.

$\boxed{\supseteq}$ suppose $h, h_1 \in H$. Then, $h_1 = h(h^{-1}h_1h)h^{-1} \in hHh^{-1}$. Thus, if $h_1 \in H$, then $h_1 \in hHh^{-1}$. So, $H \subseteq hHh^{-1}$.

You can try to find an example of where this fails if $H \nleq G$.

(6) (b) $H \leq C_G(H)$ iff H is abelian.

($\Longrightarrow$) Suppose $H \leq C_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}$.
Then, $h'h = hh'$ for all $h' \in H$. So, H is abelian.

($\Longleftarrow$) Suppose H is abelian. Then $h'h = hh'$ for all $h, h' \in H$. Hence
$$H \leq C_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}.$$
∎

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(7)

(b) This follows from §1.2 #4.

(a) Suppose $n = 2k+1$ is odd. Then $r^n = 1$ and $r^l \neq 1$ if $l < n$. Note that if $0 < i < n$, then
$r^i s = s r^{n-i} \neq s r^i$ because if so $n - i = i$
which would imply that $n = 2i$ was even.
So, $\boxed{\text{none of}}$ $r, r^2, r^3, \dots, r^{n-1}$ are in $Z(D_{2n})$.

Similarly, $\boxed{\text{if } 0<i<n}$ $(sr^i)s = s^2 r^{-i} = r^{n-i} \neq r^i = s(sr^i)$.
for the same reason ($r^{n-i} = r^i$ iff $n - i = i$ iff n is even)

So $\boxed{\text{none of}}$ $s, sr, sr^2, \dots, sr^{n-1}$ are in $Z(D_{2n})$.

Hence $Z(D_{2n}) = \{1\}$.

(11) $Z(G) \leq N_G(A)$ for any subset A of G

Let $z \in Z(G)$. Then, $zAz^{-1} = \{zaz^{-1} \mid a \in A\}$

$$= \{zz^{-1}a \mid a \in A\} = \{a \mid a \in A\} = A.$$

since $z \in Z(G)$

So, $z \in N_G(A)$. $\blacksquare$