3.1

1. We first note that $\varphi^{-1}(E)$ is non-empty. Why? $1_H \in E$ and $\varphi(1_G) = 1_H$ since φ is a homomorphism. Hence $1_G \in \varphi^{-1}(E)$.

Let $x, y \in \varphi^{-1}(E)$. Then by def. we have that $\varphi(x) = e_1$ and $\varphi(y) = e_2$ where $e_1, e_2 \in E$. Since E is a subgroup, $e_2^{-1} \in E$. Thus,

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = e_1 e_2^{-1} \in E.$$

So, $xy^{-1} \in \varphi^{-1}(E)$. Thus, $\varphi^{-1}(E) \leq G$.

Now suppose that $E \trianglelefteq H$. Let $g \in G$ and $x \in \varphi^{-1}(E)$. Then $\varphi(x) = e$ where $e \in E$. Note that $\varphi(gxg^{-1}) = \varphi(g)\, e\, \varphi(g)^{-1}$ which is in E since $E \trianglelefteq H$ and $\varphi(g) \in H$. Hence, $gxg^{-1} \in \varphi^{-1}(E)$. Thus, $g\varphi^{-1}(E)g^{-1} \subseteq \varphi^{-1}(E)$. So, $\varphi^{-1}(E)$ is normal.

$\ker(\varphi) = \varphi^{-1}(\{1_H\})$. So, $\ker(\varphi)$ is normal.

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3) Let $x, y \in A/B$. Then $x = a_1B$ and $y = a_2B$ where $a_1, a_2 \in A$. Thus,

$$xy = (a_1B)(a_2B) \underset{\substack{\text{def of} \\ \text{coset operation}}}{=} (a_1a_2)B \underset{\substack{\text{since } A \\ \text{is abelian}}}{=} (a_2a_1)B = (a_2B)(a_1B) = yx.$$

So, A/B is abelian.

<u>Ex:</u> S_3 contains A_3. A_3 is normal in S_3 and $S_3/A_3 \cong \mathbb{Z}_2$ which is abelian. We'll discuss this more in class when we get to A_3.

You can probably also use D_6 and $\langle r \rangle$. Try it.

(3,1) (4) case 1: $\alpha \geq 0$.

We prove this by induction. If $\alpha = 0$, then $(gN)^0 = N = g^0 N$. Suppose that $(gN)^k = g^k N$ for some integer k with $k \geq 0$.

Then

$$(gN)^{k+1} = (gN)^k (gN) = (g^k N)(gN) = g^{k+1} N.$$

By def of group operation in G/N

By induction, $(gN)^\alpha = g^\alpha N$ for $\alpha \geq 0$.

case 2: $\alpha < 0$

Note that $(g^{-1}N)(gN) = (g^{-1}g)N = N$

and $(gN)(g^{-1}N) = (gg^{-1})N = N$

Hence, $(gN)^{-1} = g^{-1}N$.

Now suppose $\alpha < 0$. Then,

$$(gN)^\alpha = ((gN)^{-1})^{-\alpha} = (g^{-1}N)^{-\alpha}$$

$$= (g^{-1})^{-\alpha} N = g^\alpha N.$$

by case 1

3.1

22(a) By 2.1 #10, $H \cap K$ is a subgroup of G. Let $g \in G$ and $x \in H \cap K$.
Since $x \in H$ and H is normal, $gxg^{-1} \in H$.
Since $x \in K$ and K is normal, $gxg^{-1} \in K$.

Hence; $gxg^{-1} \in H \cap K$.

Thus, $g(H \cap K)g^{-1} \subseteq H \cap K$.

So, $H \cap K$ is a normal subgroup of G.

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36 Suppose that $G/Z(G)$ is cyclic.
Then, $G/Z(G) = \langle xZ(G) \rangle$ for some $x \in G$.
Let $a, b \in G$. Since every coset in $G/Z(G)$ is of the form $x^i Z(G)$ for some $i \in \mathbb{Z}$ we have that $a \in x^k Z(G)$ and $b \in x^l Z(G)$ for some $k, l \in \mathbb{Z}$ (because the cosets of $Z(G)$ partition the group G). So, $a = x^k z_a$ and $b = x^l z_b$ for some $z_a, z_b \in Z(G)$. Thus,

$$ab = x^k z_a x^l z_b = x^k x^l z_a z_b = x^{k+l} z_b z_a$$
$$= x^{l+k} z_b z_a = x^l x^k z_b z_a = x^l z_b x^k z_a = ba.$$

3.1 $\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(\bar{0},\bar{0}), (\bar{1},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{2}), (\bar{0},\bar{3}), (\bar{1},\bar{3})$

A) $\langle(\bar{0},\bar{1})\rangle = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3})\}$

Cosets:

$(\bar{0},\bar{0}) + \langle(\bar{0},\bar{1})\rangle = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3})\}$

$(\bar{1},\bar{0}) + \langle(\bar{0},\bar{1})\rangle = \{(\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}), (\bar{1},\bar{3})\}$

So, $\mathbb{Z}_2 \times \mathbb{Z}_4 / \langle(\bar{0},\bar{1})\rangle = \{(\bar{0},\bar{0}) + \langle(\bar{0},\bar{1})\rangle, (\bar{1},\bar{0}) + \langle(\bar{0},\bar{1})\rangle$

B) $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$\langle s\rangle = \{1, s\}$

$r\langle s\rangle = \{r, rs\} = \{r, sr^{-1}\} = \{r, sr^3\}$

$r^2\langle s\rangle = \{r^2, r^2 s\} = \{r^2, sr^{-2}\} = \{r^2, sr^2\}$

$r^3\langle s\rangle = \{r^3, r^3 s\} = \{r^3, sr^{-3}\} = \{r^3, sr\}$

So, $D_8/\langle s\rangle = \{\langle s\rangle, r\langle s\rangle, r^2\langle s\rangle, r^3\langle s\rangle\}$

3.1

c)

$\langle(\bar1,\bar1)\rangle = \{(\bar0,\bar0), (\bar1,\bar1), (\bar2,\bar2), (\bar0,\bar3), (\bar1,\bar4), (\bar2,\bar5)\}$

$(\bar1,\bar0) + \langle(\bar1,\bar1)\rangle = \{(\bar1,\bar0), (\bar2,\bar1), (\bar0,\bar2), (\bar1,\bar3), (\bar2,\bar4), (\bar0,\bar5)\}$

$(\bar0,\bar1) + \langle(\bar1,\bar1)\rangle = \{(\bar0,\bar1), (\bar1,\bar2), (\bar2,\bar3), (\bar0,\bar4), (\bar1,\bar5), (\bar2,\bar0)\}$

~~So,~~ $\cancel{\mathbb{Z}_3\times\mathbb{Z}_6/\langle(\bar1,\bar1)\rangle = \{(\bar0,\bar0)+\langle(\bar1,\bar1)\rangle, \langle(\bar1,\bar0)\rangle}$

Let $H = \langle(\bar1,\bar1)\rangle$.

Then, $\mathbb{Z}_3\times\mathbb{Z}_6/H = \{(\bar0,\bar0)+H, (\bar1,\bar0)+H, (\bar0,\bar1)+H\}$.

$(\bar2,\bar1)+H = (\bar1,\bar0)+H \neq (\bar0,\bar0)+H$

$[(\bar2,\bar1)+H]+[(\bar2,\bar1)+H] = (\bar4,\bar2)+H = (\bar1,\bar2)+H = (\bar0,\bar1)+H$
$\neq (\bar0,\bar0)+H$

$[(\bar2,\bar1)+H]+[(\bar2,\bar1)+H]+[(\bar2,\bar1)+H] = (\bar6,\bar3)+H$
$= (\bar0,\bar3)+H = H$

So, $(\bar2,\bar1)+H$ has order 3 in $\mathbb{Z}_3\times\mathbb{Z}_6/\langle(\bar1,\bar1)\rangle$.