$\boxed{4.3}$

$\boxed{2(a)}$ $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

conjugacy class of 1 is $\boxed{\{1\}}$

conjugacy class of r is

$$D_8 \cdot r = \{1r1^{-1}, rrr^{-1}, r^2rr^{-2}, r^3rr^{-3}, srs^{-1}, (sr)r(sr)^{-1}, (sr^2)r(sr^2)^{-1}, (sr^3)r(sr^3)^{-1}\}$$

$$= \{r, r, r, r, r^3, r^3, r^3, r^3\} = \boxed{\{r, r^3\}}$$

conjugacy class of r^2 is

$$D_8 \cdot r^2 = \{1r^21, rr^2r^{-1}, r^2r^2r^{-2}, r^3r^2r^{-3}, sr^2s^{-1}, sr\,r^2(sr)^{-1}, (sr^2)r^2(sr^2)^{-1}, (sr^3)r^2(sr^3)^{-1}\}$$

$$= \{r^2, r^2, r^2, r^2, r^2, r^2, r^2, r^2\} = \boxed{\{r^2\}}$$

conjugacy class of s is

$$D_8 \cdot s = \{1s1, rsr^{-1}, r^2sr^{-2}, r^3sr^{-3}, sss^{-1}, (sr)s(sr)^{-1}, (sr^2)s(sr^2)^{-1}, (sr^3)s(sr^3)^{-1}\}$$

$$= \{s, sr^2, s, sr^2, s, sr^2, s, sr^2\} = \boxed{\{s, sr^2\}}$$

conjugacy class of sr is

$$D_8 \cdot sr = \{ 1 \cdot sr 1, rsrr^{-1}, r^2 sr r^{-2}, r^3 sr r^{-3}, s sr s^{-1}, sr\, sr(sr)^{-1}, sr^2 sr(sr^2)^{-1}, sr^3 sr (sr^3)^{-1} \}$$

$$= \{ sr, sr^3, sr, sr^3, sr^3, sr, sr, sr^3 \}$$

$$= \boxed{\{ sr, sr^3 \}}$$

So, the conjugacy classes are

$\{1\}$, $\{r, r^3\}$, $\{r^2\}$, $\{s, sr^2\}$, $\{sr, sr^3\}$.

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4) Let $S \subseteq G$ and $g \in G$.

(a) Prove that $g N_G(S) g^{-1} = N_G(gSg^{-1})$

(b) Prove that $g C_G(S) g^{-1} = C_G(gSg^{-1})$

(a) Let $x \in N_G(S)$. Then, $xSx^{-1} = S$. Let $s \in S$.

Then $(gxg^{-1})\, gsg^{-1} (gxg^{-1})^{-1} = gxg^{-1} g s g^{-1} g x^{-1} g^{-1} = gxsx^{-1}g^{-1}$

$= gs'g^{-1}$ for some $s' \in S$ since $xsx^{-1} \in S$. Hence,

$gxg^{-1} \in N_G(gSg^{-1})$.

~~Conversely, let $y \in N_G(gSg^{-1})$. Then, given $ygsg^{-1}y^{-1} \in S$ for every $s \in S$. Let $s \in S$. Then,~~

~~$ygsg^{-1}y^{-1} = s'$ for some $s' \in S$.~~

~~Hence~~

Conversely,

Let $y \in N_G(gSg^{-1})$. Then, $y(gSg^{-1})y^{-1} = gSg^{-1}$.

Hence, $(g^{-1}yg)S(g^{-1}yg)^{-1} = g^{-1}(ygSg^{-1}y^{-1})g = g^{-1}(gSg^{-1})g$
$= S$.

Then, $g^{-1}yg \in N_G(S)$. So, $y \in gN_G(S)g^{-1}$.

(b) You try this one.

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(8) $Z(S_n) = \{1\}$ for all $n \geq 3$.

Let $\sigma \in S_n$ with $\sigma \neq$ identity. Then, $\sigma(a) = b$ and $\sigma(c) = d$ for some $1 \leq a, b, c, d \leq n$ with $a \neq b$, $c \neq d$, $a \neq c$, and $b \neq d$.

Let $\tau = (b, d)$.

Then we have

$$(\sigma \circ \tau)(a) = \sigma(\tau(a)) = \sigma(a) = b$$
$$(\sigma \circ \tau)(c) = \sigma(\tau(c)) = \sigma(c) = d$$
$$(\tau \circ \sigma)(a) = \tau(\sigma(a)) = \tau(b) = d$$
$$(\tau \circ \sigma)(c) = \tau(\sigma(c)) = \tau(d) = b$$

Hence, $\sigma \circ \tau = \tau \circ \sigma$. So, $\sigma \notin Z(S_n)$. Thus, $Z(S_n) = \{1\}$.

11 (a) $\sigma_1 = (1,2)(3,4,5)$
$\sigma_2 = (1,2,3)(4,5)$

Let $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$.

Then, $\tau \sigma_1 \tau^{-1} = \sigma_2$. (See Prop. 10)

11 (b) $\sigma_1 = (1,5)(3,7,2)(10,6,8,11)$
$\sigma_2 = (3,7,5,10)(4,9)(13,11,2)$

Let $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 2 & 8 & 1 & 9 & 7 & 11 & 5 & 6 & 3 & 10 & 12 \end{pmatrix}$

Then, $\tau \sigma_1 \tau^{-1} = \sigma_2$. (See Prop 10)