4.5

13 Let $|G| = 56 = 2 \cdot 28 = 2^3 \cdot 7$

Let P be a ~~...~~ Sylow 2-subgroup of G and Q be a Sylow 7-subgroup of G.

Then $|P| = 2^3 = 8$ and $|Q| = 7$.

So, $n_2 \equiv 1 \pmod 2$ and n_2 divides 7 and $n_7 \equiv 1 \pmod 7$ and n_7 divides $2^3 = 8$.

The possibilities for n_2 are 1 and 7. The possibilities for n_7 are 1 and 8.

Let's show that we cannot have both $n_2 = 7$ and $n_7 = 8$ at the same time. This will show that either $n_2 = 1$ or $n_7 = 1$ which shows that either $P \trianglelefteq G$ or $Q \trianglelefteq G$.

Suppose that $n_2 = 7$ and $n_7 = 8$. Thus there are 7 2-sylow subgroups, call them $P_1, P_2, P_3, P_4, P_5, P_6$, and P_7. And there are 8 7-Sylow subgroups, call them $Q_1, Q_2, \ldots, Q_8$.

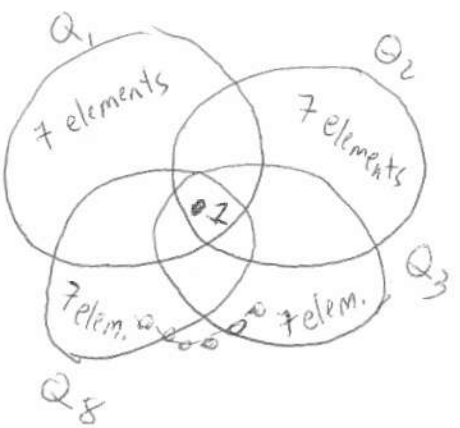
If $i \neq j$, then $Q_i \cap Q_j \leq Q_i$ and $Q_i \cap Q_j \leq Q_j$ which gives that ~~...~~ $Q_i \cap Q_j = \{1\}$ because by Lagrange $|Q_i \cap Q_j| = 1$ or 7 and cannot equal pg. 2

7 because if it did then $Q_i \cap Q_j = Q_i$ and $Q_i \cap Q_j = Q_j$ which would give that $Q_i = Q_j$.

Thus, $Q_i \cap Q_j = \{1\}$ for all $i \neq j$.

Therefore the number of distinct elements in $Q_1 \cup Q_2 \cup \dots \cup Q_8$ is

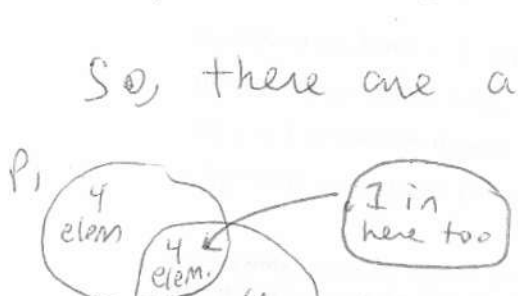
$$8 \cdot [7-1] + 1 = 49.$$



Note that $|P_i \cap Q_j| = \{1\}$ since $|P_i \cap Q_j|$ divides $|P_i| = 8$ and $|P_i \cap Q_j|$ divides $|Q_j| = 7$.

Consider P_1 and P_2. If $P_1 \neq P_2$, then $|P_1 \cap P_2| = 1, 2,$ or 4 (since if $|P_1 \cap P_2| = 8$, then $P_1 = P_2$.)

So, there are at least $4+4+4-1 = 12-1 = 11$ elements in $P_1 \cup P_2$ that are not in $Q_1 \cup \dots \cup Q_8$ is.
(we subtract 1 because $1 \in P_1 \cup P_2$)



Hence $|P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup \dots \cup Q_8| \geq 49 + 11 = 60 > 56 = |G|$. Thus, $n_2 = 1$ or $n_7 = 1$. ∎