$\boxed{9.4}$

(1) (a) $f(x) = x^2 + x + \bar{1}$ in $\mathbb{Z}_2[x]$

$$\left.\begin{array}{l} f(\bar{0}) = \bar{1} \\ f(\bar{1}) = \bar{1} \end{array}\right\}$$ Since $\deg(f) = 2 \leq 3$ and f has no roots in $\mathbb{Z}_2$, f is irreducible over $\mathbb{Z}_2$.

(b) $f(x) = x^3 + x + \bar{1}$ in $\mathbb{Z}_3[x]$

$f(\bar{0}) = \bar{1}$

$f(\bar{1}) = \bar{0} \leftarrow$

$$\begin{array}{r|l}
 & x^2 + x + \bar{2} \\ \hline
\underbrace{x + \bar{2}}_{x - \bar{1}} & x^3 + x + \bar{1} \\
 & -(x^3 + \bar{2}x^2) \\ \hline
 & x^2 + x + \bar{1} \\
 & -(x^2 + \bar{2}x) \\ \hline
 & \bar{2}x + \bar{1} \\
 & -(\bar{2}x + \bar{1}) \\ \hline
 & \bar{0}
\end{array}$$

So, $f(x) = (x + \bar{2})(x^2 + x + \bar{2})$

Let $g(x) = x^2 + x + \bar{2}$.

$$\left.\begin{array}{l} g(\bar{0}) = \bar{2} \\ g(\bar{1}) = \bar{4} = \bar{1} \\ g(\bar{2}) = \bar{8} = \bar{2} \end{array}\right\}$$ g is irreducible over $\mathbb{Z}_3$ since $\deg(g) \leq 3$ and g has no roots in $\mathbb{Z}_3$.

② (a) $f(x) = x^4 - 4x^3 + 6$

f is irreducible using $p=2$ and Eisenstein's criteria since $2 \nmid 1$, $2 \mid -4$, $2 \mid 6$, but $2^2 \nmid 6$.

(b) $f(x) = x^6 + 30x^5 - 15x^3 + 6x - 120$

Note: $120 = 2 \cdot 60 = 2^2 \cdot 30 = 2^4 \cdot 15 = 2^4 \cdot 3 \cdot 5$

f is irreducible using $p=3$ and Eisenstein's criteria since $3 \nmid 1$, $3 \mid 30$, $3 \mid -15$, $3 \mid 6$, $3 \mid -120$, $3^2 \nmid -120$.

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⑥ (a) $9 = 3^2$.

Let $g(x) = x^2 + x + \bar{2}$.

By 1(b), g is irreducible over $\mathbb{Z}_3$.

Let $\mathbb{F}_9 = \mathbb{Z}_3[x]/(x^2+x+\bar{2})$.

Let $I = (x^2+x+\bar{2})$. In $\mathbb{F}_9$, $x^2 + I = -x - \bar{2} + I$
$= \bar{2}x + \bar{1} + I$.

Let $\theta = x + I$. Then,

$\mathbb{F}_9 = \left\{ \bar{a} + \bar{b}\theta \;\middle|\; \begin{matrix} \theta^2 = \bar{2}\theta + \bar{1} \\ \bar{a}, \bar{b} \in \mathbb{Z}_3 \end{matrix} \right\}$ ← Here we are identifying $\bar{a}$ with $\bar{a} + I$ and $\bar{b}$ with $\bar{b} + I$.