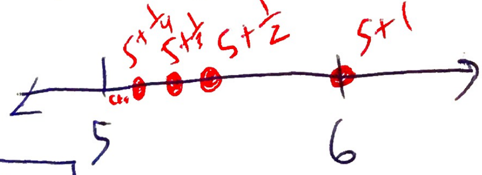
Hw #1 Solutions

① (a) $X = \left\{ 5 + \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$

$= \left\{ 5+1, 5+\frac{1}{2}, 5+\frac{1}{3}, 5+\frac{1}{4}, \ldots \right\}$

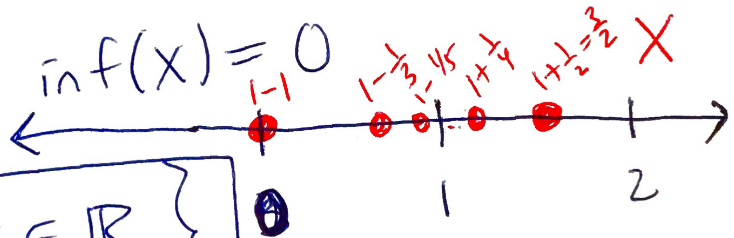
$\sup(X) = 6$, $\inf(X) = 5$



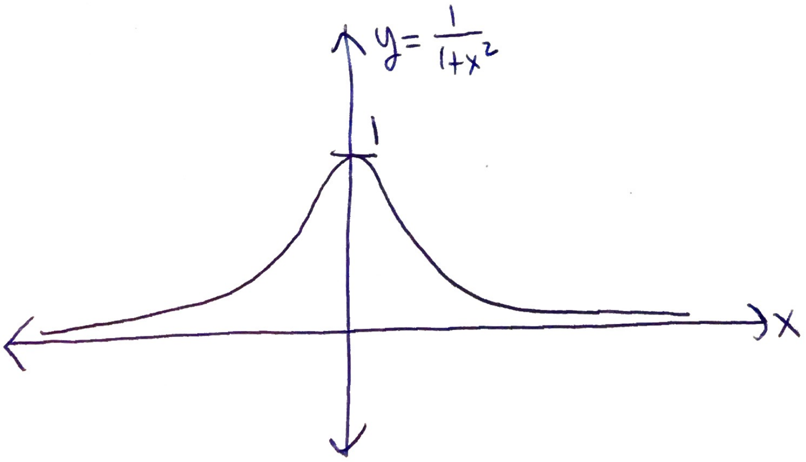
(b) $X = \left\{ 1 + \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{N} \right\}$

$= \left\{ 1 - \frac{1}{1}, 1 + \frac{1}{2}, 1 - \frac{1}{3}, 1 + \frac{1}{4}, \ldots \right\}$

$\sup(X) = \frac{3}{2}$ $\inf(X) = 0$



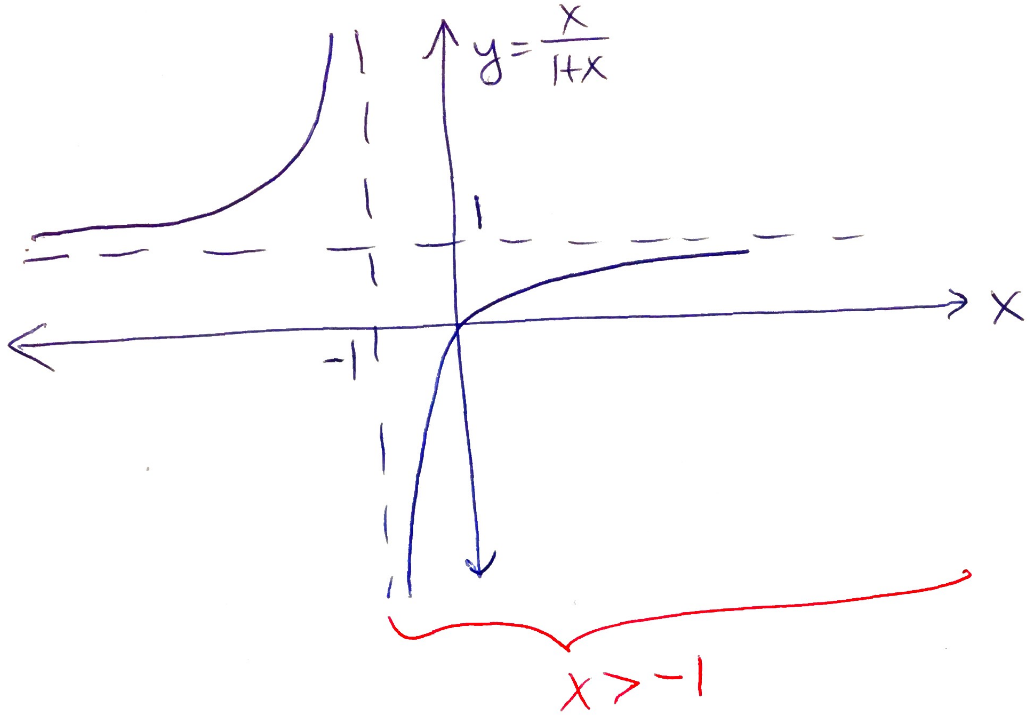
(c) $X = \left\{ \frac{1}{1+x^2} \;\middle|\; x \in \mathbb{R} \right\}$



$\sup(X) = 1$

$\inf(X) = 0$

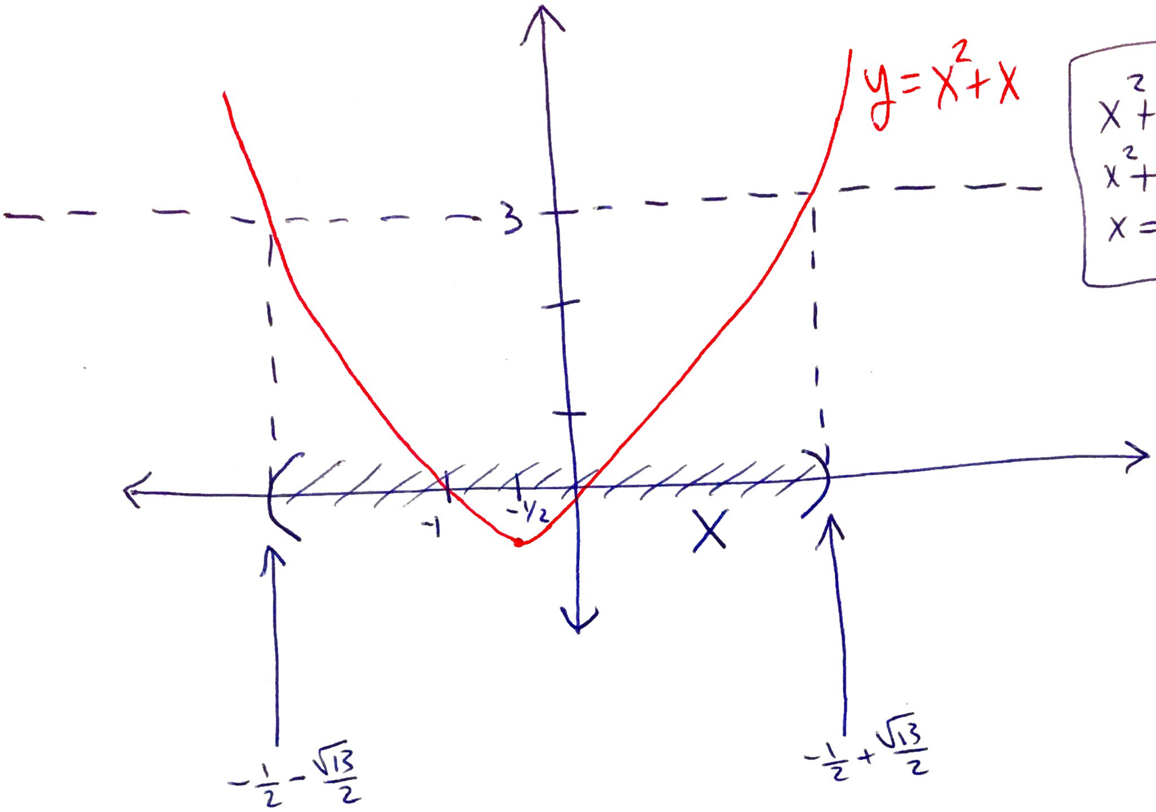
(d) $X = \left\{ \frac{x}{1+x} \,\middle|\, x \in \mathbb{R},\ x > -1 \right\}$



$\sup(X) = 1$

$\inf(X)$ does not exist

(e) $X = \{x \in \mathbb{R} \mid x^2 + x < 3\}$

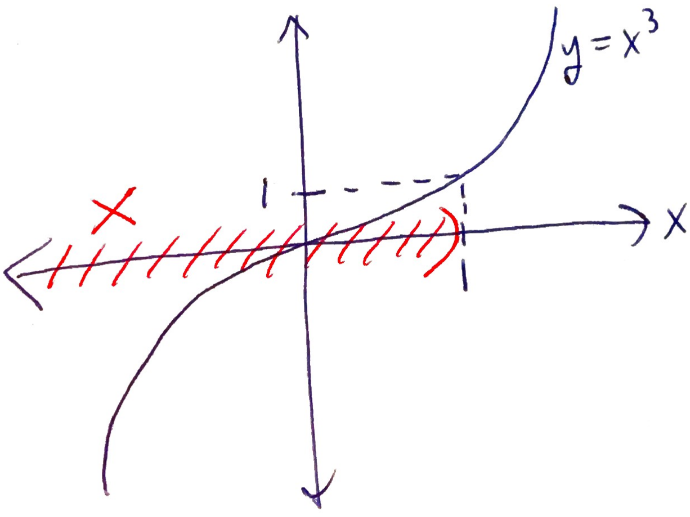


$$\boxed{\begin{aligned} x^2 + x &= 3 \\ x^2 + x - 3 &= 0 \\ x &= -\frac{1}{2} \pm \frac{\sqrt{13}}{2} \end{aligned}}$$

$\inf(X) = -\frac{1}{2} - \frac{\sqrt{13}}{2}$

$\sup(X) = -\frac{1}{2} + \frac{\sqrt{13}}{2}$

(f) $X = \{x \in \mathbb{R} \mid x^3 < 1\}$



$\inf(X)$ does not exist

$\sup(X) = 1$

(g) $X = \{.3, .33, .333, .3333, .33333, \ldots\}$

Note that $\frac{1}{3} = 0.33333333333333\ldots$

$\inf(X) = .3$

$\sup(X) = \frac{1}{3}$

~~(Pf by contradiction)~~

~~② Suppose that $x > 0$.~~

~~Let $\varepsilon = \frac{x}{2}$.~~

~~Then $\varepsilon > 0$ since $x > 0$.~~

~~[illegible] By assumption of the problem [illegible] $\varepsilon < x$ in this instance, [illegible] $\leq \varepsilon$. But [illegible] Contradiction. Thus, $x = 0$.~~

~~③ Suppose that a and b are both supremums for S. Then a and b are both upper bounds for S.~~

~~Since a is a supremum for S and b is an upper bound for S, by def of supremum we have that $a \leq b$.~~

~~Since b is a supremum for S and a is an upper bound for S, by def of supremum we have that $b \leq a$. Hence $a = b$.~~

② (proof by contradiction)

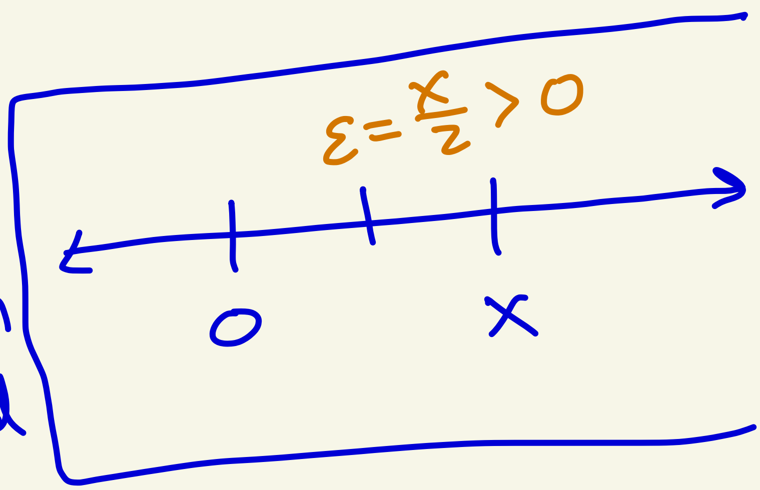
Let $x \in \mathbb{R}$ with $x \geq 0$.

Suppose also that $x \leq \varepsilon$ for every $\varepsilon > 0$.

We will show that $x = 0$.

Suppose instead that $x > 0$.

Then $\varepsilon = \frac{x}{2} > 0$.



But by assumption then we would have $x \leq \varepsilon$.

But then $x \leq \frac{x}{2}$.

This implies that $\frac{x}{2} \leq 0$.

This gives $x \leq 0$ which contradicts $x > 0$.

Hence $x > 0$ can't be the case and so $x = 0$. ▨

(3) Suppose a and b are both supremums of S.

Thus, a and b are both upper bounds for S.

Since a is a supremum for S and b is an upper bound for S, by def of supremum, we have that $a \leq b$.

Since b is a supremum for S and a is an upper bound for S, by def of supremum, we have that $b \leq a$.

Since $a \leq b$ and $b \leq a$ we have that $a = b$. $\blacksquare$

4) We are given that b is an upper bound for S and that $b \in S$.

Let's show that $b = \sup(S)$.

(i) We already have that b is an upper bound for S.

(ii) Let's show that b is the least upper bound for S.

Let c be another upper bound for S.

Then, $x \leq c$ for all $x \in S$.

Since $b \in S$ this gives $b \leq c$.

Thus, b is the least upper bound for S.

By (i) and (ii), $b = \sup(S)$. $\blacksquare$

5 (a)

$\boxed{\text{Proof 1}}$ – This method uses the def of supremum

We are given that $a = \sup(A)$ and $b = \sup(B)$ exist.

We are also given that $A \cap B \neq \emptyset$.

Claim: $A \cap B$ is bounded from above

Note that $x \leq a$ for all $x \in A$ because a is an upper bound for A.

Since $A \cap B \subseteq A$ this means that $x \leq a$ for all $x \in A \cap B$.

Hence a is an upper bound for $A \cap B$.

$\boxed{\text{Claim}}$

Similarly one can show that b is an upper bound for $A \cap B$.

Thus, $x \leq a$ for all $x \in A \cap B$
and $x \leq b$ for all $x \in A \cap B$.

Therefore, if $c = \min\{a, b\}$ then $x \leq c$ for all $x \in A \cap B$.

So, $c = \min\{a, b\}$
$= \min\{\sup(A), \sup(B)\}$
is an upper bound for $A \cap B$.

Because $A \cap B$ is bounded from above we know that $\sup(A \cap B)$ exists.

Since $\sup(A \cap B)$ is the supremum of $A \cap B$ and c is an upper bound for $A \cap B$, by the def of supremum we have that $\sup(A \cap B) \leq c$.

[supremum is the least upper bound]

Thus,

$$\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}.$$

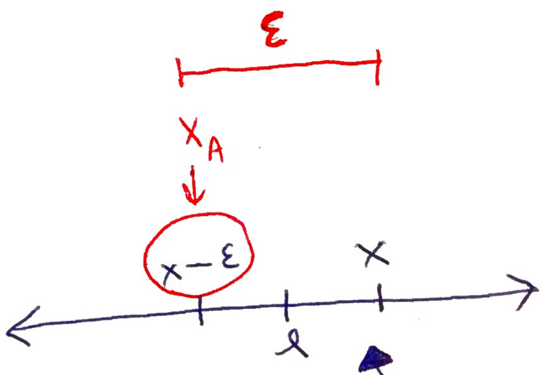
$\blacksquare$

~~(4) It is given to us that b is an upper bound for S. We must show that b is the least upper bound for S. Suppose that x is another upper bound for S. Then $s \le x$ for all $s \in S$. But $b \in S$. Thus, $b \le x$. So, $b = \sup(S)$.~~

Proof 2 — This method uses the useful sup/inf fact

(5) (a) $\boxed{\text{TRUE}}$

Since $A \cap B \subseteq A$ and A is bounded from above, thus we know that $A \cap B$ is bounded from above. Hence $x = \sup(A \cap B)$ exists.



Let $x_A = \sup(A)$.

Let's show that $x \le x_A$.

Suppose that $x > x_A$.

Then $x - x_A > 0$.

Let $\varepsilon = x - x_A$.

Since x is the supremum of $A \cap B$, there exists $l \in A \cap B$ such that $x - \varepsilon < l \le x$.

So, $x_A < l \le x$ (since $x - \varepsilon = x_A$).

This contradicts the fact that $\sup(A) = x_A$ since $l \in A$.

Similarly you can show that if $x_B = \sup(B)$ then $x \le x_B$.

So, $x \le \min\{x_A, x_B\}$ ▨

5 (b) $\boxed{\text{FALSE}}$

Let $A = [5, 27) \cup (-1, 3]$

$B = (-1, 5)$.

Then $A \cap B = (-1, 3]$

<u>Note:</u> $\sup(A) = 27$

$\sup(B) = 5$

$\sup(A \cap B) = 3$

~~Another example is on the next page.~~

<u>Another example:</u>

$A = (1, 5)$

$B = [1, 5]$

$\inf(A) = \inf(B) = 1$

$\sup(A) = \sup(B) = 5$

But $A \neq B$

⑤ (c) $\boxed{\text{True}}$

$\boxed{\text{Proof 1}}$ – This method uses the def of sup

Let $a = \sup(A)$ and $b = \sup(B)$.
We will assume for this proof that

$$a \leq b.$$

If we assumed $b \leq a$ the same proof would work with a & b interchanged.

Since $a \leq b$ we have that

$$b = \max\{\overbrace{\sup(A)}^{a}, \overbrace{\sup(B)}^{b}\}.$$

<u>Claim 1: b is an upper bound for $A \cup B$</u>

Let $x \in A \cup B$.

If $x \in A$, then $x \leq a \leq b$.

since $a = \sup(A)$

If $x \in B$, then $x \leq b$.

since $b = \sup(B)$

Thus no matter the case we have that $x \le b$.

Thus, b is an upper bound for $\sup(A \cup B)$.

$\boxed{\text{Claim 1}}$

<u>Claim 2: b is the least upper bound for $A \cup B$</u>

Suppose c is another upper bound for $A \cup B$.

Then, $x \le c$ for all $x \in A \cup B$.

This implies that both

$x \le c$ for all $x \in A$

and $x \le c$ for all $x \in B$

Thus c is an upper bound for A
and c is an upper bound for B.

Thus, $a \le c$ and $b \le c$
by def of supremum and
since $a = \sup(A)$ and $b = \sup(B)$.

Thus, $\max\{a, b\} \le c$.

Thus, $b \le c$.
So, b is the least upper bound for $A \cup B$.

$\boxed{\text{Claim 2}}$

By claim 1 and claim 2,
$\sup(A \cup B) = b = \max\{\underbrace{\sup(A)}_{a}, \underbrace{\sup(B)}_{b}\}$ $\blacksquare$

$\boxed{\text{Proof 2}}$ — This proof is slightly different than proof 1 in that it uses the useful sup/inf fact

5 (c) $\boxed{\text{True}}$

Let $x_A = \sup(A)$ and $x_B = \sup(B)$.
Without loss of generality, assume that $x_B \leq x_A$ (the same proof will work if $x_A \leq x_B$ by interchanging A and B).

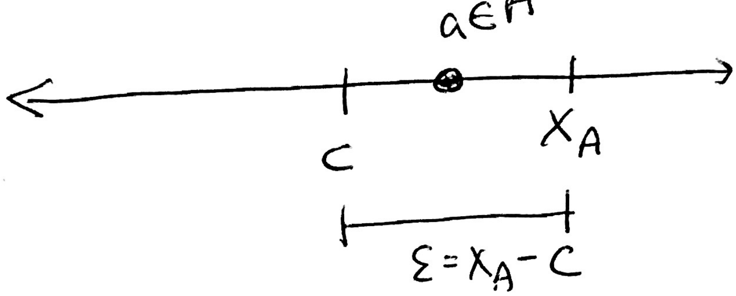
We want to show that x_A is the supremum of $A \cup B$.

(i) First off, if $l \in A \cup B$, then $l \in A$ or $l \in B$.
If $l \in A$, then $l \leq x_A$ since x_A is an upper bound for A.
If $l \in B$, then $l \leq x_B \leq x_A$.
Therefore, $l \leq x_A$ in either case.
So, x_A is an upper bound for $A \cup B$.

(ii) We now show that x_A is the least upper bound for $A \cup B$.
Let c be another upper bound of $A \cup B$.
We want to show that $x_A \leq c$.
We do this by showing that $c < x_A$ is impossible.

Suppose that $c < x_A$.
Then $0 < x_A - c$.

Let $\varepsilon = x_A - c > 0$.



By the useful sup/inf fact, since $x_A = \sup(A)$ we know that there exists $a \in A$ with $x_A - \varepsilon < a \leq x_A$.

Thus, since $x_A - \varepsilon = x_A - (x_A - c) = c$, we have that $c < a \leq x_A$.

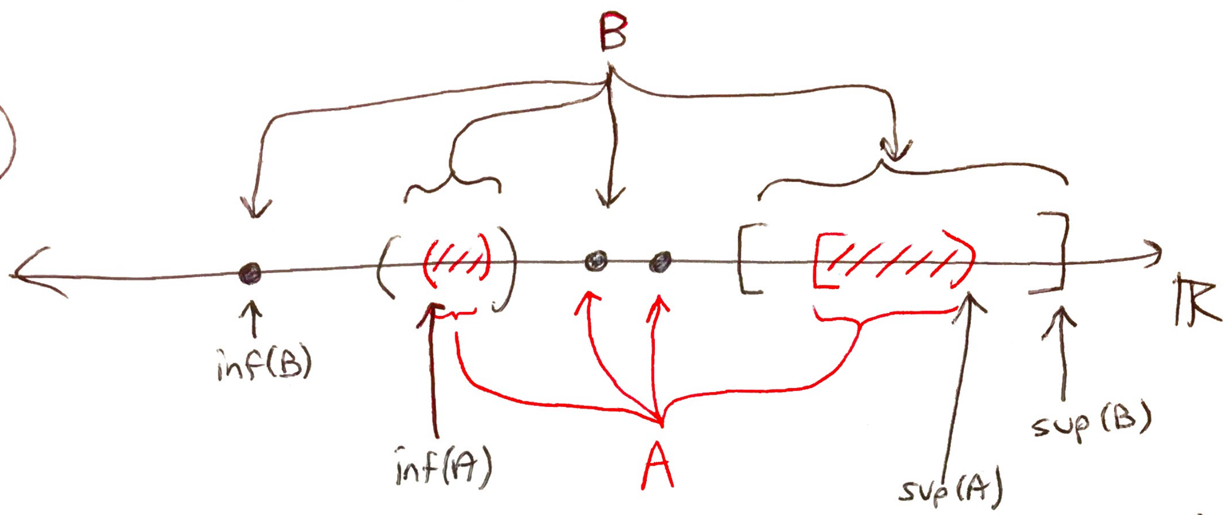
But $a \in A \cup B$.

The equation $c < a$ contradicts the fact that c is an upper bound for $A \cup B$.

Hence $x_A \leq c$.

We have shown that $x_A = \sup(A \cup B)$. ∎

6



(picture for $A \subseteq B$. Not necessarily how A is, but can use to help you think about it.)

(a) Suppose $A \subseteq B$.

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

Since s_B is the supremum of B we know that it is an upper bound for B.

Hence $b \le s_B$ for all $b \in B$.

Since $A \subseteq B$ this means that $a \le s_B$ for all $a \in A$ too.

Thus, s_B is an upper bound for A.

Since $s_A = \sup(A)$ is the least upper bound on A and s_B is an upper bound on A we know that $s_A \le s_B$. So, $\sup(A) \le \sup(B)$.

The proof that $\inf(A) \ge \inf(B)$ is the same idea as $\sup(A) \le \sup(B)$.

Try to do this part on your own.

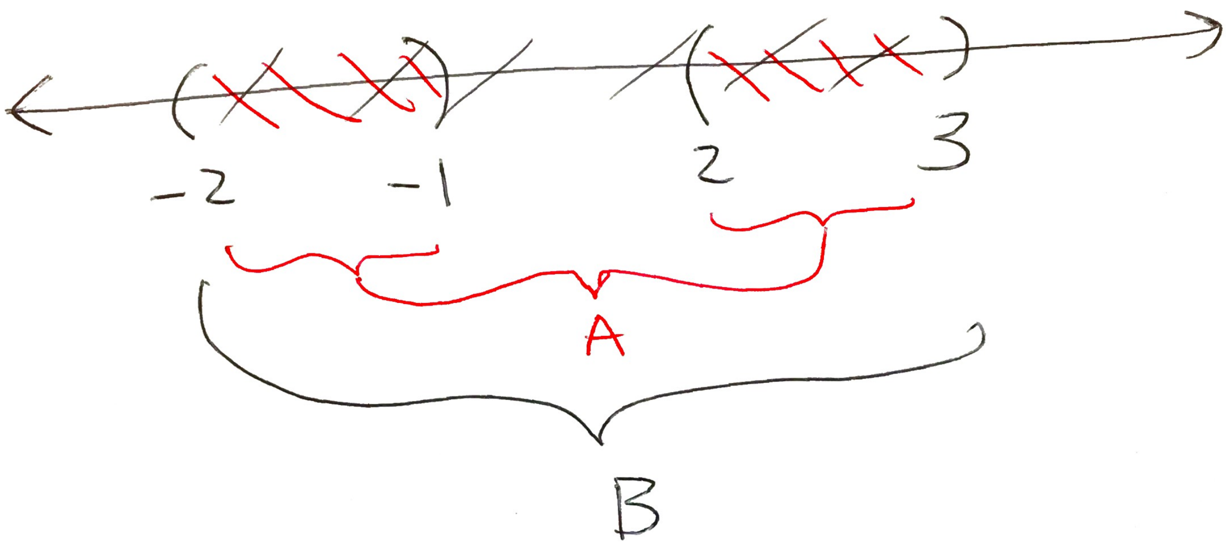
(Just flip the $\le$ to $\ge$.)

(b) False.

Heres an example.

Let $A = (-2,-1) \cup (2,3)$

$B = (-2,3)$.



$\inf(A) = -2 = \inf(B)$

$\sup(A) = 3 = \sup(B)$

but $A \neq B$.

For the next problem, number 7, the main fact that is used is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

7 (a) Suppose that $a<x<b$ and $a<y<b$.
We want to show that $|x-y|<b-a$.
We break the proof into two cases.

Case 1: Suppose $x \geq y$.

Then $|x-y| = x-y$ (since $x-y \geq 0$).

We know $a<x<b$.
So, by ~~[illegible]~~ adding $-a$ through the equation we get $0<x-a<b-a$.

Now $a<y$ is given.
So, $-a>-y$.

Thus, $x-a>x-y$.
So, $x-y<x-a<b-a$.

Thus, $|x-y|=x-y<b-a$.

RECALL

$$|u| = \begin{cases} u, & \text{if } u \geq 0 \\ -u, & \text{if } u<0 \end{cases}$$

DEF OF ABS. VALUE

Case 2: Suppose $x<y$.

Then, $x-y<0$. So, $|x-y|=-(x-y)=y-x$.
We know $a<y<b$. Subtracting a we get $0<y-a<b-a$.
Now $a<x$ is given. So, $-a>-x$. So, $y-a>y-x$.
Thus, $|x-y|=y-x<y-a<b-a$.

$\boxed{}$ 7(a)

(7) (b) We break this into cases.

Case 1: $a \leq b$

Then $a - b \leq 0$.
So, $|a-b| = -(a-b) = b-a$.
Also, $b - a \geq 0$.
So, $|b-a| = b-a$.
Thus, $|a-b| = |b-a|$

case 2: $a > b$

Then $a-b > 0$.
So, $|a-b| = a-b$.
Also, $b-a < 0$.
So, $|b-a| = -(b-a) = a-b$.
Thus, $|a-b| = |b-a|$.

(7) (c) We break this into cases.

Case 1: $a \geq 0$ and $b \geq 0$.

Then $|a| = a$ and $|b| = b$.
Since $ab \geq 0$ we have $|ab| = ab$.
Hence $|ab| = |a| \cdot |b|$

Case 2: $a \geq 0$ and $b < 0$

Then $|a| = a$ and $|b| = -b$.
Since $ab \leq 0$, we have that $|ab| = -ab$.
Hence $|ab| = |a| \cdot |b|$.

Case 3: $a < 0$ and $b \geq 0$

Then $|a| = -a$ and $|b| = b$.
Since $ab \leq 0$, we have that $|ab| = -ab$.
Hence $|ab| = |a| \cdot |b|$.

Case 4: $a < 0$ and $b < 0$

Then $|a| = -a$ and $|b| = -b$.
Since $ab > 0$, we have that $|ab| = ab$.
Thus, $|ab| = |a| \cdot |b|$.

7 (d)

Note that

$$|a| = |(a-b)+b| \leq |a-b| + |b|$$

triangle inequality

So, $\boxed{|a| - |b| \leq |a-b|.}$

Also, $|b| = |(b-a)+a| \leq |b-a| + |a|$

So, $\cancel{|b| - |a| \leq |b-a|,}$ $-|b-a| \leq |a| - |b|.$

Recall from (b) that $|a-b| = |b-a|.$

So, $\boxed{-|a-b| = -|b-a| \leq |a| - |b|,}$

Therefore,

$$\boxed{-|a-b| \leq |a| - |b| \leq |a-b|,}$$

Recall that if $c \geq 0$, then $|x| \leq c$ iff $-c \leq x \leq c.$

In our situation $c = |a-b|$ and $x = |a| - |b|.$

Thus, $\big||a| - |b|\big| \leq |a-b|.$