Math 4570 - Homework # 1

Vector spaces and subspaces

1. Are the following vector spaces? Prove or disprove.

   (a) Let

   $$V = C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{ f is continuous on all of } \mathbb{R}\}$$

   and $F = \mathbb{R}$. For example, $f_1(x) = x^2$ and $f_2(x) = \sin(x)$ are in V. Vector addition is the usual function addition defined as $(f + g)(x) = f(x) + g(x)$ and scalar multiplication is the usual defined as $(\alpha f)(x) = \alpha \cdot f(x)$

   (b) $V = \mathbb{R}^2$ and $F = \mathbb{R}$ where vector addition is defined as $(x, y) + (a, b) = (x + a, y + b)$ and scalar multiplication is defined as
   $\alpha \odot (x, y) = (2\alpha x, 2\alpha y)$.
   [Here I changed the notation for the scalar multiplication to $\odot$ since it is different from the usual scalar multiplication.]

   (c) V is the set of positive real numbers, that is $V = \{x \in \mathbb{R} | x > 0\}$ and $F = \mathbb{R}$ where vector addition is defined as $x \oplus y = xy$ and scalar multiplication is defined as $\alpha \odot x = x^\alpha$.
   [Here I changed the notation to $\oplus$ and $\odot$ to reflect that we are defining a new plus and scalar multiplication that is different from the usual one.]

2. Are the following sets W subspaces of the vector spaces V ? Prove or disprove.

   (a) $V = \mathbb{R}^3$, $F = \mathbb{R}$, $W = \{(a, b, c) \mid a = 3b \text{ and } c = -b\}$

   (b) $V = \mathbb{C}^3$, $F = \mathbb{C}$, $W = \{(a, b, c) \mid a = c + 2\}$

   (c) $V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$, $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b + c + d = 0 \right\}$

   (d) $V = M_{2,2}(\mathbb{R})$, $F = \mathbb{R}$, $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 0 \right\}$

   (e) $V = P_3(\mathbb{R})$, $F = \mathbb{R}$, $W = \{a + bx + cx^2 + dx^3 \mid a = 0\}$

(f) $V = P_3(\mathbb{C})$, $F = \mathbb{C}$, $W = \{a + bx \mid a + b = i\}$

3. Let F be a field. Prove the following.

   (a) Let 0_1 and 0_2 be in F where $0_1 + a = a + 0_1 = a$ and $0_2 + a = a + 0_2 = a$ for all $a \in F$. Show that $0_1 = 0_2$.

   (b) Let 1_1 and 1_2 be in F where $1_1 \cdot a = a \cdot 1_1 = a$ and $1_2 \cdot a = a \cdot 1_2 = a$ for all $a \in F$. Show that $1_1 = 1_2$.

   (c) Let $a \in F$ and $d_1, d_2 \in F$. If $a + d_1 = d_1 + a = 0$ and $a + d_2 = d_2 + a = 0$, then $d_1 = d_2$.

   (d) Let $b \in F$ with $b \neq 0$ and $f_1, f_2 \in F$. If $b \cdot f_1 = f_1 \cdot b = 1$ and $b \cdot f_2 = f_2 \cdot b = 1$, then $f_1 = f_2$.

4. Let V be a vector space over a field F. Prove the following.

   (a) For each $x \in V$ we have that $0x = \mathbf{0}$. Here 0 is the additive identity of F and $\mathbf{0}$ is the additive identity of V.

   (b) For each $a \in F$ and $x \in V$ we have that $(-a)x = -(ax) = a(-x)$.

   (c) For each $a \in F$ we have that $a\mathbf{0} = \mathbf{0}$ where $\mathbf{0}$ is the additive identity of V.

5. Let V be a vector space of a field F. Let W be a subset of V. Then W is a subspace of V if and only if the following three conditions hold.

   (a) $\mathbf{0} \in W$

   (b) If $x, y \in W$ then $x + y \in W$.

   (c) If $c \in F$ and $x \in W$ then $cx \in W$.

6. Let V be a vector space over a field F. Let W_1 and W_2 be subspaces of V. Prove that $W_1 \cap W_2$ is a subspace of V.

7. Let V be a vector space over a field F. Let W_1 and W_2 be subspaces of V. Define the **sum** of W_1 and W_2 to be the set

$$W_1 + W_2 = \{x + y \mid x \in W_1 \text{ and } y \in W_2\}$$

   (a) Prove that $W_1 \subset W_1 + W_2$ and $W_2 \subset W_1 + W_2$.

   (b) Prove that $W_1 + W_2$ is a subspace of V.

   (c) Prove that if W is subspace of V that contains both W_1 and W_2 then W also contains $W_1 + W_2$.