Math 5680

Homework # 1

Series

1. Determine whether or not the following series converge. If the series converges, what does it converge to?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{2^{n-1}}$

   (b) $\displaystyle\sum_{n=3}^{\infty} \frac{e+1}{2^n \pi^{n+3}}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{10^{n+1}}{2^n \sqrt{3}^{\,n+3}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+i)^n}{5+(1+i)^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

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2. Let $n_0 \geq 1$ be an integer. Show that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=n_0}^{\infty} a_n$ converges. Here the a_n are complex numbers.

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3. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two convergent sequences of complex numbers.

   (a) If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

   (b) If $\sum_{k=1}^{\infty} a_k = A$ and $\alpha \in \mathbb{C}$, then $\sum_{k=1}^{\infty}(\alpha \cdot a_k) = \alpha A$

4. (Cauchy Criterion for series) Let $\sum_{k=1}^{\infty} a_k$ be a series of complex numbers. Prove: $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an $N > 0$ such that if $n \geq N$ then

$$\left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon$$

for all $p = 1, 2, 3, 4, \ldots$.

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5. (Comparison Test) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be sequences of positive real numbers. Suppose further that $0 < a_k \leq b_k$ for all k.

   (a) Prove: If $\sum b_k$ converges then $\sum a_k$ converges.

   (b) Prove: If $\sum a_k$ diverges then $\sum b_k$ diverges.

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6. Determine whether or not the following series converges. Does it converge absolutely?

   (a) $\displaystyle\sum_{n=1}^{\infty} \sin(\pi i^n)$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + (-i)^n}{n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} z^n$ where $z \in \mathbb{C}$ and $|z| < 1$

   (d) $\displaystyle\sum_{n=1}^{\infty} z^n$ where $z \in \mathbb{C}$ and $|z| \geq 1$

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7. Let $\sum_{k=1}^{\infty} a_k$ be a series of complex numbers. Prove: If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

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8. (a) Prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   [Note: This proof is a pretty tricky, so don't get frustrated if you can't do it without looking at the solution. It's a classic proof, which is why I put it in here.]

   (b) Let p be a real number with $p \leq 1$. Prove that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

   [Hint: Compare it to the harmonic series.]
