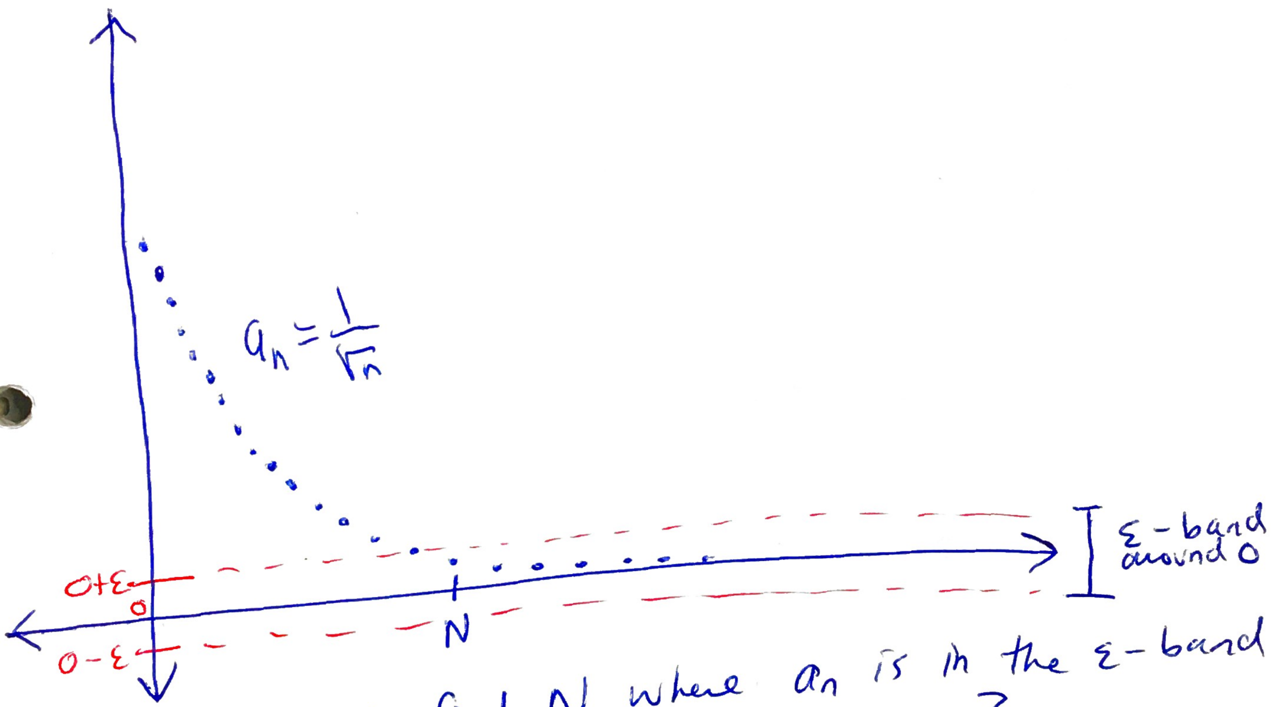
HW #2 Solutions

1

Suppose $\varepsilon > 0$ is fixed,



We want to find N where a_n is in the ε-band if $n \geq N$. When is $\frac{1}{\sqrt{n}} < 0 + \varepsilon$? When $\frac{1}{\varepsilon} < \sqrt{n}$. Or when $\frac{1}{\varepsilon^2} < n$.

Let N be an integer with $\frac{1}{\varepsilon^2} < N$.

Then if $n \geq N > \frac{1}{\varepsilon^2}$ we have that $\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \varepsilon$.

ε	$1/\varepsilon^2$	Pick N greater than
0.01	10,000	10,000
0.001	1,000,000	1,000,000
0.0001	100,000,000	100,000,000

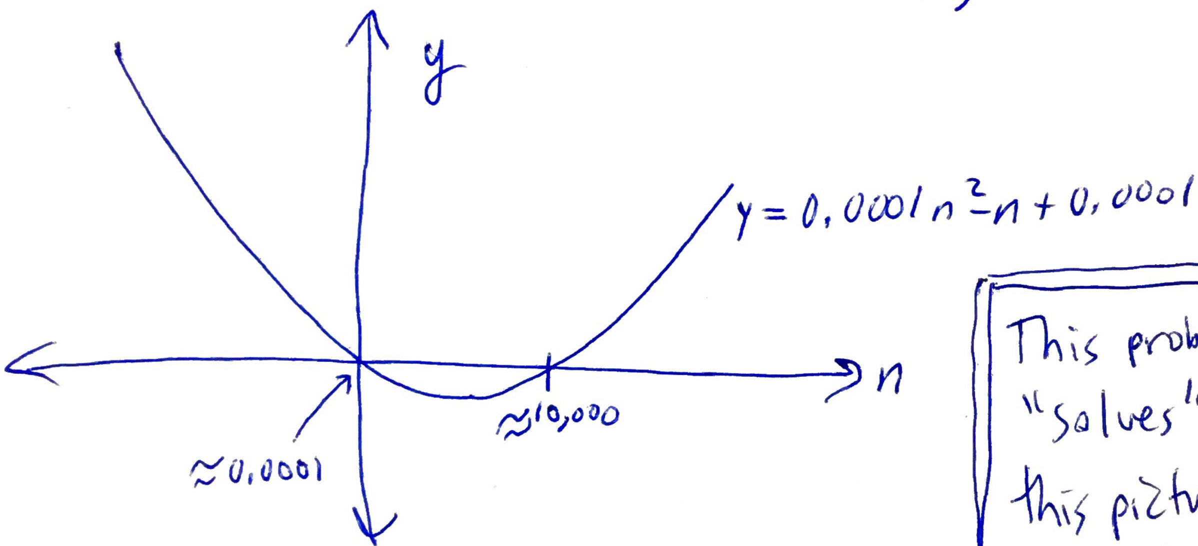
(2) To make $\left|\frac{n}{n^2+1} - 0\right| < 0.0001$

we need $\frac{n}{n^2+1} < 0.0001$. (since $\frac{n}{n^2+1} > 0$ we know $\left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1}$)

This is equivalent to $n < 0.0001n^2 + 0.0001$

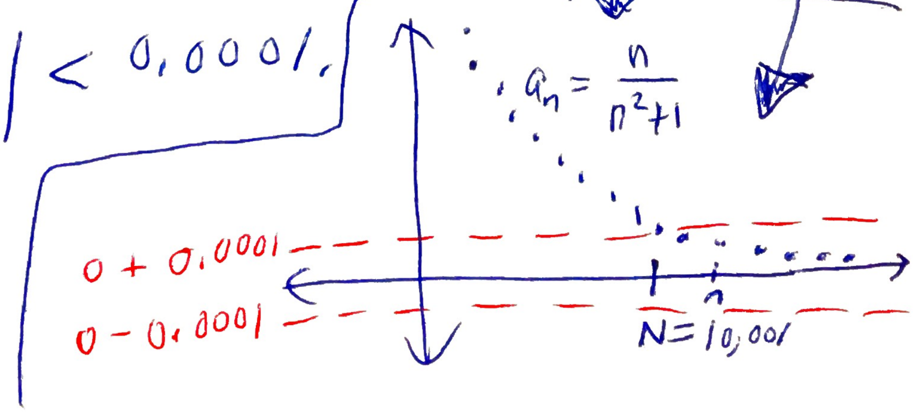
or $0 < 0.0001n^2 - n + 0.0001$.

The zeros of $0 = 0.0001n^2 - n + 0.0001$ are $n \approx 0.0001$ and $n \approx 10{,}000$.



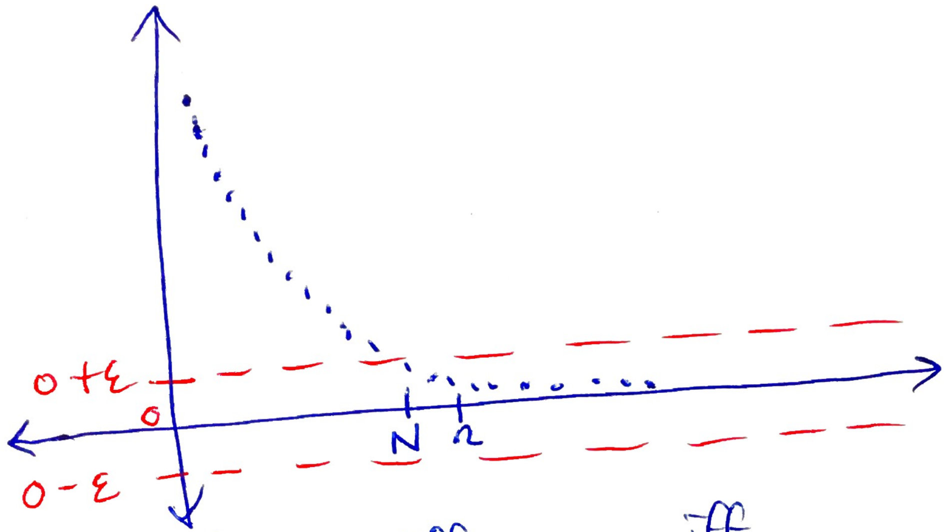
This problem "solves" this picture

So, if $N = 10{,}001$ and if $n \geq N$ then $0 < 0.0001n^2 - n + 0.0001$ or $\left|\frac{n}{n^2+1} - 0\right| < 0.0001$.



3 (a)

Let $\varepsilon > 0$ be fixed.
We want to find $N > 0$ so that if $n \geq N$ then $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$.



Note that $\left|\frac{1}{\sqrt{n}}\right| < \varepsilon$ iff $\frac{1}{\sqrt{n}} < \varepsilon$ iff $\frac{1}{\varepsilon} < \sqrt{n}$ iff $\frac{1}{\varepsilon^2} < n$.

Suppose $N > \frac{1}{\varepsilon^2}$. Then if $n \geq N > \frac{1}{\varepsilon^2}$ we have that $\left|\frac{1}{\sqrt{n}} - 0\right| = \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} < \varepsilon$.

---

3 (b) Let $\varepsilon > 0$ be fixed.
We want to find $N > 0$ so that if $n \geq N$ then $\left|\frac{n+2}{5n-3} - \frac{1}{5}\right| < \varepsilon$.

Note that $\left|\frac{n+2}{5n-3} - \frac{1}{5}\right| < \varepsilon$ iff $\left|\frac{5n+10-5n+3}{5(5n-3)}\right| < \varepsilon$

iff $\left|\frac{13}{25n-15}\right| < \varepsilon$ $\underbrace{\text{iff}}_{}$ $\frac{13}{25n-15} < \varepsilon$ iff $\frac{13}{\varepsilon} < 25n-15$

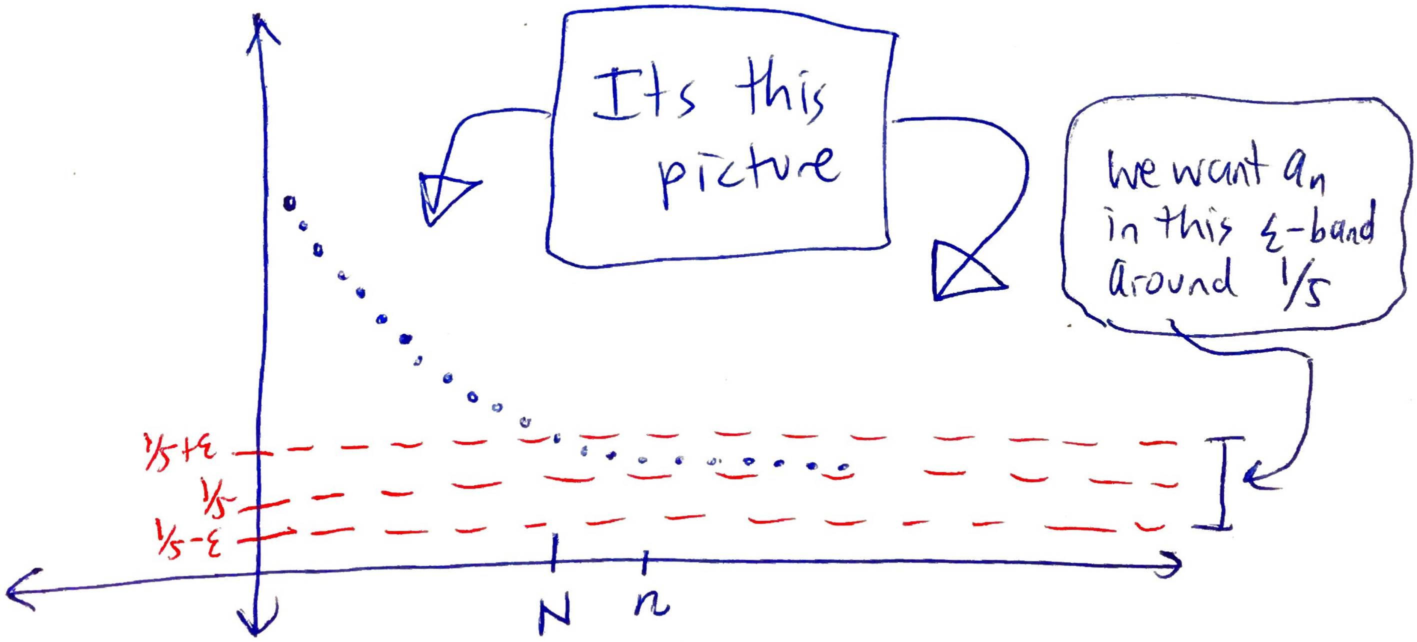
$\boxed{\frac{13}{25n-15} > 0 \text{ if } n \geq 1}$

iff $\frac{13}{\varepsilon} + 15 < 25n$ iff $\dfrac{\frac{13}{\varepsilon}+15}{25} < n$.

Let N be an integer with $N > \dfrac{\frac{13}{\varepsilon}+15}{25}$.

Then if $n \geq N$, then $n > \dfrac{\frac{13}{\varepsilon}+15}{25}$.

So, if $n \geq N$, then $\left|\frac{n+2}{5n-3} - \frac{1}{5}\right| < \varepsilon$.



③ (c) Let $\varepsilon > 0$ be fixed.

$\sqrt{n+1} - \sqrt{n} > 0$ since $\sqrt{n+1} > \sqrt{n}$

Note that

$$\left|\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right| = \sqrt{n+1} - \sqrt{n}$$

$$= \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}$$

$\boxed{\sqrt{n+1} > \sqrt{n}}$

So, $\left|\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right| < \frac{1}{2\sqrt{n}}$.

We want to find N so that if $n \geq N$, then $\frac{1}{2\sqrt{n}} < \varepsilon$.

Note that $\frac{1}{2\sqrt{n}} < \varepsilon$ iff $\frac{1}{\varepsilon} < 2\sqrt{n}$

iff $\frac{1}{2\varepsilon} < \sqrt{n}$ iff $\frac{1}{4\varepsilon^2} < n$.

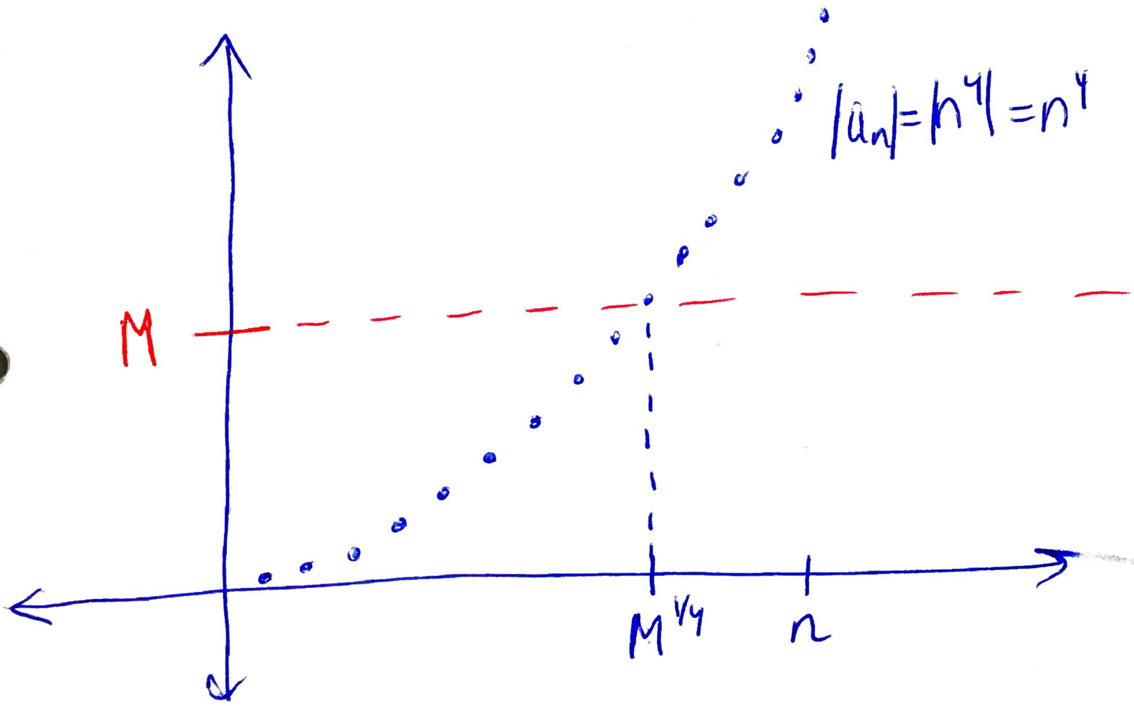
Let $N > \frac{1}{4\varepsilon^2}$.

If $n \geq N > \frac{1}{4\varepsilon^2}$, then from above

$$\left|\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right| < \frac{1}{2\sqrt{n}} < \varepsilon.$$

③ (d) We prove this by showing that (n^4) is unbounded. [Sequences with limits are bounded.]

Let $M > 0$ be an arbitrary fixed number. Suppose that $n > M^{1/4}$. Then $|n^4| = n^4 > (M^{1/4})^4 = M$. So, (n^4) is unbounded.



This picture can be solved for any $M > 0$. Hence (n^4) is unbounded.

③ (e) Let $\varepsilon > 0$ be fixed.

We need to find $N > 0$ so that if $n \geq N$, then $\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon$.

Note that

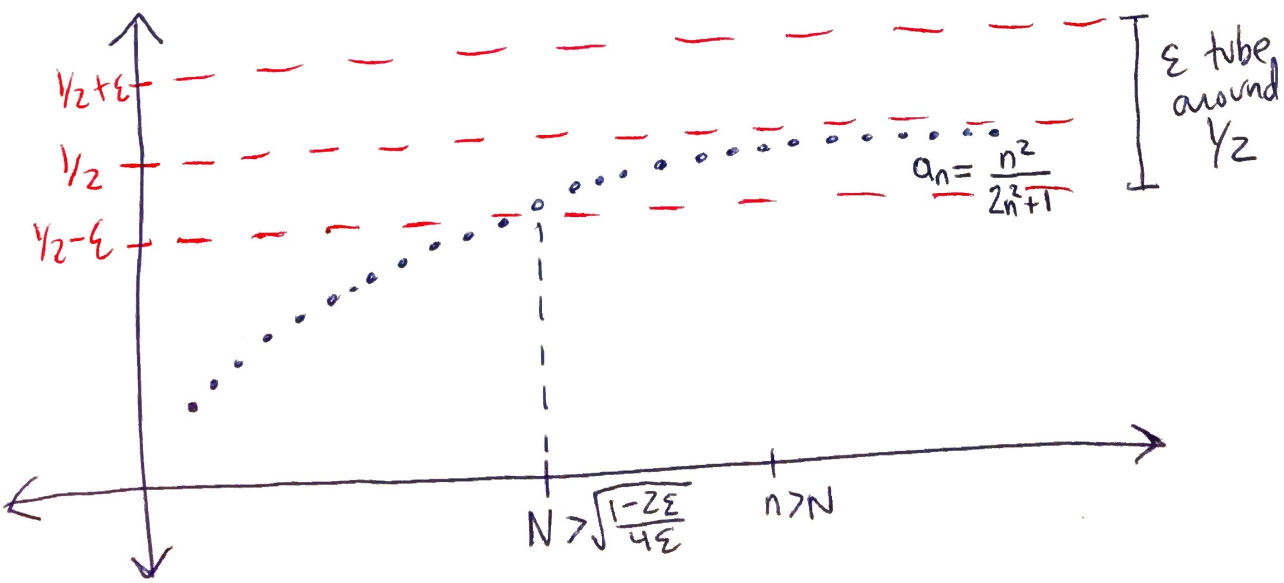
$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| = \left| \frac{2n^2 - 2n^2 - 1}{2(2n^2+1)} \right| = \left| \frac{-1}{4n^2+2} \right| = \frac{1}{4n^2+2}$$

Note that $\frac{1}{4n^2+2} < \varepsilon$ iff $1 < 4\varepsilon n^2 + 2\varepsilon$

iff $\frac{1-2\varepsilon}{4\varepsilon} < n^2$ iff $\sqrt{\frac{1-2\varepsilon}{4\varepsilon}} < n$.

Let $N > \sqrt{\frac{1-2\varepsilon}{4\varepsilon}}$. Then if $n \geq N$, we have

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon.$$



③ (f) Note that

$$\left|\frac{\sqrt{n^2+1}}{n!} - 0\right| = \frac{\sqrt{n^2+1}}{n!} \leq \frac{\sqrt{n^2+n^2}}{n!} = \frac{\sqrt{2n^2}}{n!}$$

$$= \frac{\sqrt{2}\cdot n}{n!} = \frac{\sqrt{2}}{(n-1)!} \leq \frac{\sqrt{2}}{n-1}$$

$\boxed{1 \leq n^2}$

$\boxed{\frac{\sqrt{n^2+1}}{n!} > 0}$

$\boxed{n! = n\cdot[(n-1)!]}$

ex: $6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 6[5!]$

$(n-1)! \geq n-1$ so, $\frac{1}{(n-1)!} \leq \frac{1}{n-1}$

Let $\varepsilon > 0$. Choose N such that $N > \frac{\sqrt{2}}{\varepsilon} + 1$.

Then $n \geq N$ gives us

$$\left|\frac{\sqrt{n^2+1}}{n!} - 0\right| \leq \frac{\sqrt{2}}{n-1} \leq \frac{\sqrt{2}}{N-1} < \frac{\sqrt{2}}{\left(\frac{\sqrt{2}}{\varepsilon}+1\right)-1} = \varepsilon.$$

$n \geq N \rightarrow n-1 \geq N-1 \rightarrow \frac{1}{n-1} \leq \frac{1}{N-1}$

---

③ (g) Let $\varepsilon > 0$.

Note that $\left|\frac{1}{2^n} - 0\right| < \varepsilon$ iff $\frac{1}{2^n} < \varepsilon$ iff $\frac{1}{\varepsilon} < 2^n$

iff $\log\left(\frac{1}{\varepsilon}\right) < \log(2^n)$ iff $\log\left(\frac{1}{\varepsilon}\right) < n\log(2)$

iff $\frac{\log\left(\frac{1}{\varepsilon}\right)}{\log(2)} < n$. Let $N > \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log(2)}$.

Then from above, if $n \geq N$, we have that

$\left|\frac{1}{2^n} - 0\right| < \varepsilon$.

③ (h) Let's show that $\left(\frac{n^2}{n+1}\right)$ is an unbounded sequence. Thus, it cannot converge.

Let $M > 1$ be a fixed number.

Note that $\left|\frac{n^2}{n+1}\right| > M$ iff $\frac{n^2}{n+1} > M$

iff $n^2 > nM + M$ iff $n^2 - Mn > M$ iff $n(n-M) > M$.

Suppose $n > 2M$.

Then $n(n-M) > 2M(2M-M) = 2M^2 > M$.

$\boxed{\text{since } M>1}$

Thus, if $n > 2M$, then $\left|\frac{n^2}{n+1}\right| > M$.

So, since M was arbitrary, the sequence

$\frac{n^2}{n+1}$ is unbounded.

3(i) We show that $(-n^2+1)$ is an unbounded sequence. Let $M > 0$ be a fixed number.

Note that $\left|1-n^2\right| = \left|n^2-1\right| = n^2-1$.

$\boxed{|-a|=|a|}$ $\boxed{n^2-1 \geq 0 \text{ if } n \geq 1}$

And $n^2-1 > M$ iff $n^2 > M+1$ iff $n > \sqrt{M+1}$.

So, if $n > \sqrt{M+1}$, then $\left|1-n^2\right| > M$.

Thus, the sequence $(1-n^2)$ is unbounded.

④ Suppose that $\lim_{n\to\infty} a_n = A$ and $\alpha \in \mathbb{R}$.
Let $\varepsilon > 0$ be fixed. We want to find N so that if $n \geq N$, then $|\alpha a_n - \alpha A| < \varepsilon$.

<u>case 1: $\alpha = 0$</u>

Then $|\alpha a_n - \alpha A| = |0 - 0| = 0 < \varepsilon$ for all $n \geq 1$.

<u>case 2: $\alpha \neq 0$.</u>

Since $\lim_{n\to\infty} a_n = A$, there exists $N > 0$ so that if $n \geq N$, then $|a_n - A| < \frac{\varepsilon}{|\alpha|}$.

Thus, if $n \geq N$, then $|\alpha a_n - \alpha A| = |\alpha||a_n - A|$
$< |\alpha| \cdot \frac{\varepsilon}{|\alpha|} = \varepsilon$. $\boxed{/\!/}$

---

⑤ We may assume that $\alpha \neq 0$ and $\beta \neq 0$, otherwise its case 1 from ④ above. Let $\varepsilon > 0$ be fixed.
Since (a_n) converges to A, there exists $N_1 > 0$ so that if $n \geq N_1$, then $|a_n - A| < \frac{\varepsilon}{2|\alpha|}$.

Since (b_n) converges to B, there exists $N_2 > 0$ so that if $n \geq N_2$, then $|b_n - B| < \frac{\varepsilon}{2|\beta|}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. Thus, if $n \geq N$, then
$|(\alpha a_n + \beta b_n) - (\alpha A + \beta B)| = |(\alpha a_n - \alpha A) + (\beta b_n - \beta B)|$
$\leq |\alpha a_n - \alpha A| + |\beta b_n - \beta B| = |\alpha||a_n - A| + |\beta||b_n - B| < |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \frac{\varepsilon}{2|\beta|}$
$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\boxed{/\!/}$

⑥ Let $\varepsilon > 0$ be fixed.

Since (a_n) converges to L, there exists $N_1 > 0$ so that if $n \geq N_1$, then $|a_n - L| < \varepsilon$ or $\boxed{-\varepsilon < a_n - L < \varepsilon,}$

Since (c_n) converges to L, there exists N_2 so that if $n \geq N_2$, then $|c_n - L| < \varepsilon$ or $\boxed{-\varepsilon < c_n - L < \varepsilon.}$

Since $a_n \leq b_n \leq c_n$, we have that $a_n - L \leq b_n - L \leq c_n - L$ for all $n \geq 1$.

Thus, if $n \geq \max\{N_1, N_2\}$ we have that

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon.$$

That is, if $n \geq \max\{N_1, N_2\}$ then

$$-\varepsilon < b_n - L < \varepsilon \quad \text{or} \quad |b_n - L| < \varepsilon.$$

Therefore, $\lim\limits_{n \to \infty} b_n = L$. ▨

---

⑦ Suppose to the contrary that $A > B$.

Then $A - B > 0$. Let $\varepsilon = \dfrac{A-B}{2}$.

Since (a_n) converges to A, there exists $N_1 > 0$ so that if $n \geq N_1$ then $|a_n - A| < \varepsilon$.

Since (b_n) converges to B, there exists $N_2 > 0$ so that if $n \geq N_2$, then $|b_n - B| < \varepsilon$.

See picture on the next page for why we pick this ε.

Let $N = \max\{N_1, N_2\}$.

If $n \geq N$, then $-\varepsilon < a_n - A < \varepsilon$

and $-\varepsilon < b_n - B < \varepsilon$.

Plugging in $\varepsilon = \frac{A-B}{2}$, we get that

if $n \geq N$, then

$$\boxed{\begin{aligned} -\left(\frac{A-B}{2}\right) &< a_n - A < \frac{A-B}{2} \\ -\left(\frac{A-B}{2}\right) &< b_n - B < \frac{A-B}{2} \end{aligned}}$$

or

$$\boxed{\begin{aligned} \frac{A}{2} + \frac{B}{2} &< a_n < \frac{3A}{2} - \frac{B}{2} \\ -\frac{A}{2} + \frac{3B}{2} &< b_n < \frac{A}{2} + \frac{B}{2} \end{aligned}}$$

Thus, if $n \geq N$, then

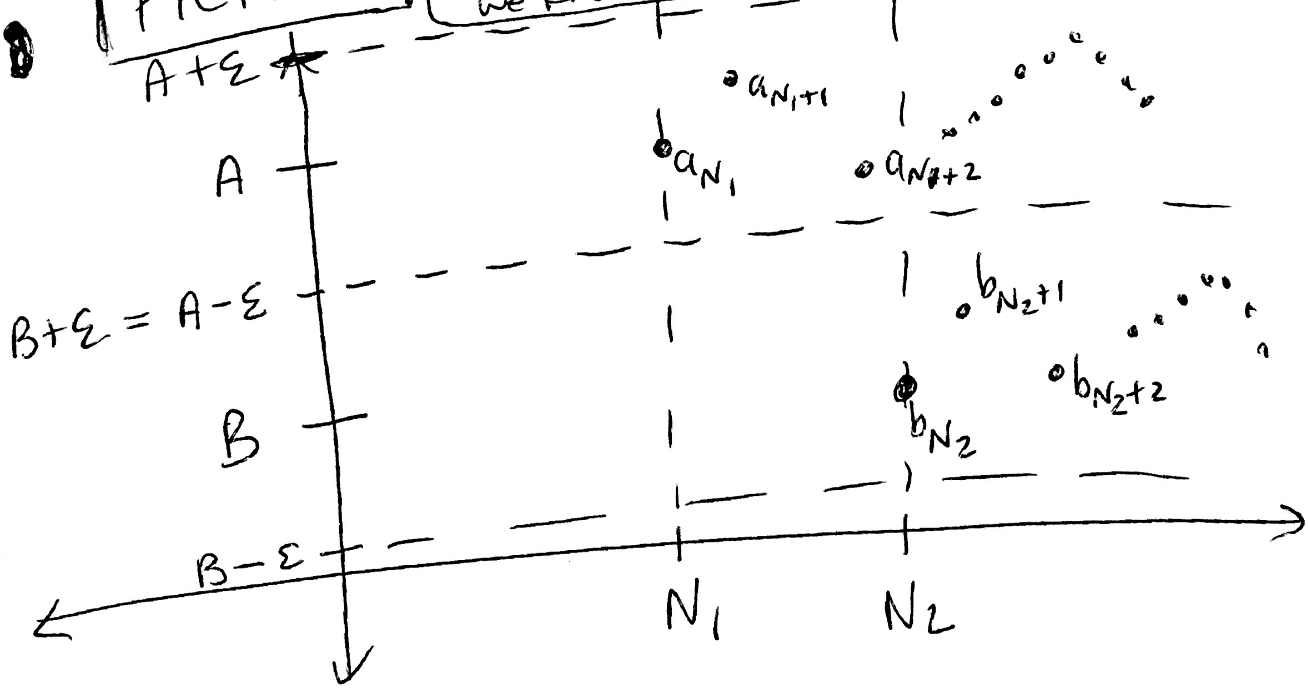
$a_n > \frac{A}{2} + \frac{B}{2} > b_n$ which is a contradiction

since $a_n \leq b_n$ for all n. $\blacksquare$

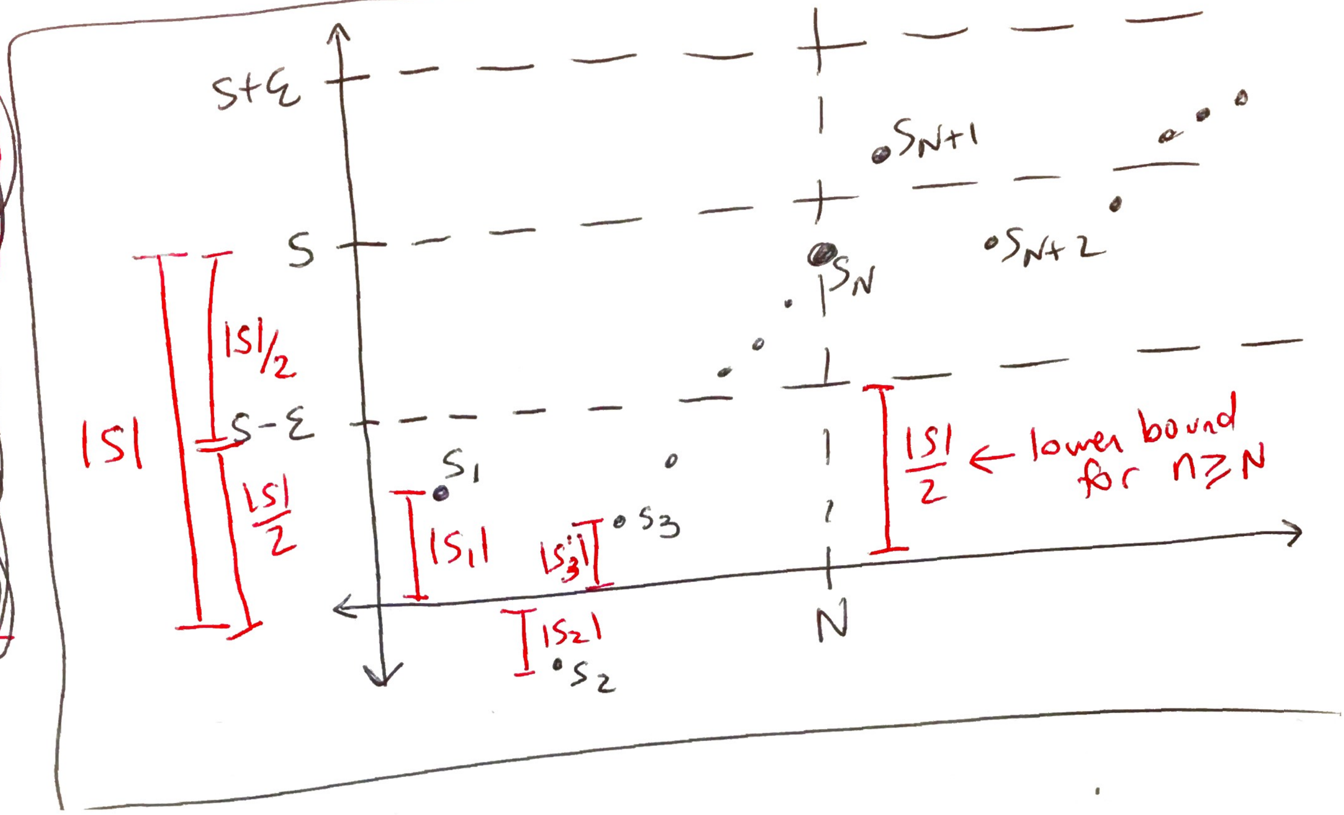
Note: I picked ε to be half the distance between A and B

PICTURE:

By doing this we force a picture like this that can't happen since we know that $a_n \leq b_n \ \forall n$,



8 (a) Let $\varepsilon = \frac{|s|}{2} > 0$ ($\varepsilon > 0$ since $s \neq 0$).



Since $\lim_{n\to\infty} s_n = s$, there exists $N > 0$ so that if $n \geq N$, then $|s_n - s| < \frac{|s|}{2} = \varepsilon$.

Note that if $n \geq N$ then

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + |s_n|$$

Hence if $n \geq N$ then $|s| < \frac{|s|}{2} + |s_n|$.

So if $n \geq N$, then $\frac{|s|}{2} < |s_n|$.

So if $1 \le n \le N-1$, then $|s_n| \leqslant |s_n|$.

If $N \le n$, then $\frac{|s|}{2} < |s_n|$.

Let $M = \min\{|s_1|, |s_2|, \ldots, |s_{N-1}|, \frac{|s|}{2}\}$.

Note that $M > 0$ since $s_n \neq 0$ for all n and $s \neq 0$.

Then, $|s_n| \geqslant M > 0$ for all n. $\blacksquare$

8(b) Let $\varepsilon > 0$,

From part (a) there exists $M > 0$ for all n.

so that $|s_n| \geqslant M$

Thus,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n \cdot s}\right| = \frac{|s - s_n|}{|s_n||s|} \leqslant \frac{|s - s_n|}{M \cdot |s|}$$

since $\lim_{n \to \infty} s_n = s$, there exists $N > 0$

so that if $n \geq N$, then $|s_n - s| < \varepsilon \cdot M \cdot |s|$.

So if $n \geqslant N$, then

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s - s_n|}{M \cdot |s|} < \frac{\varepsilon \cdot M \cdot |s|}{M \cdot |s|} = \varepsilon, \quad \blacksquare$$

(9) Let $\varepsilon = 1$.

Since (a_n) is Cauchy, there exists $N > 0$ so that if $n, m \geq N$ we have $|a_n - a_m| < 1$.

So, if $n \geq N$ we have $|a_n - a_N| < 1$.

Thus, if $n \geq N$ we have $-1 < a_n - a_N < 1$.

So, if $n \geq N$ we have $a_n < a_N + 1$.

So, if $n \geq N$ we have $|a_n| < |a_N + 1| \leq |a_N| + 1$.

Triangle inequality

Let $M = \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|, |a_N| + 1\}$.

Then $|a_n| \leq M$ for all $n \geq 1$.

So, (a_n) is a bounded sequence. $\blacksquare$