Math 465 - Homework # 2
Limits of sequences

1. The sequence $\left(\frac{1}{\sqrt{n}}\right)$ has limit 0 as you will prove in the next exercise. For each of $\epsilon = 0.01, 0.001, 0.0001$ determine an integer N such that $\left|\frac{1}{\sqrt{n}} - 0\right| < \epsilon$ for all $n \geq N$.

2. Determine a value of N such that if $n \geq N$ then $\left|\frac{n}{n^2+1} - 0\right| < 0.0001$.

3. Use the definition of limit to show that the limit of the sequence exists or does not exist.

   (a) Show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

   (b) Show that $\lim_{n\to\infty} \frac{n+2}{5n-3} = \frac{1}{5}$.

   (c) Show that $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

   (d) Show that $\lim_{n\to\infty} n^4$ does not exist.

   (e) Show that $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$.

   (f) Show that $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n!} = 0$

   (g) Show that $\lim_{n\to\infty} \frac{1}{2^n} = 0$

   (h) Show that $\lim_{n\to\infty} \frac{n^2}{n+1}$ does not exist.

   (i) Show that $\lim_{n\to\infty} (-n^2+1)$ does not exist.

4. Let (a_n) be a convergent sequence that converges to A. Let α be a real number. Prove that the sequence (αa_n) converges to αA.

5. Let (a_n) and (b_n) be convergent sequences that converge to A and B, respectively. Suppose that α and β are real numbers. Prove that the sequence $(\alpha a_n + \beta b_n)$ converges to $\alpha A + \beta B$.

6. (Squeeze Theorem) Suppose that (a_n), (b_n), and (c_n) are sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all n. If both (a_n) and (c_n) both converge to L, then (b_n) converges to L.

7. Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \leq b_n$ for all n. If both (a_n) and (b_n) converge to A and B, respectively, then $A \leq B$.

8. Prove the following:

   (a) Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all n. Suppose that $\lim_{n \to \infty} s_n = s$ where $s \neq 0$. Prove that there exists $M > 0$ such that $|s_n| > M$ for all n.

   (b) Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all n. Suppose that $\lim_{n \to \infty} s_n = s$ where $s \neq 0$. Prove that $\left(\frac{1}{s_n}\right)$ converges to $\frac{1}{s}$.

9. Suppose that (a_n) is a Cauchy sequence. Using the definition of Cauchy sequence, prove that (a_n) is a bounded sequence.