Math 455

Homework # 3 - Homomorphisms

1. Are the following functions homomorphisms? Are they isomorphisms? Prove or disprove. Recall that $\mathbb{Z}$ and $\mathbb{Q}$ are groups under addition; while $\mathbb{Q}^*$ and $\mathbb{R}^*$ are groups under multiplication.

   (a) $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n) = 5n$.

   (b) $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n) = 2n - 1$.

   (c) $\phi : \mathbb{Q} \to \mathbb{Q}$ be defined by $\phi(x) = x/5$.

   (d) $\phi : \mathbb{Q}^* \to \mathbb{Q}^*$ be defined by $\phi(x) = x^2$.

   (e) $\phi : \mathbb{Q}^* \to \mathbb{Q}^*$ be defined by $\phi(x) = 3x$.

   (f) $\phi : \mathbb{R} \to \mathbb{R}^*$ be defined by $\phi(x) = e^x$.

2. Let n be an integer with $n \geq 2$. Let

$$n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\} = \{nk \mid k \in \mathbb{Z}\}.$$

   (a) Prove that $n\mathbb{Z}$ is a group under addition.

   (b) Prove that $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

3. Let G and G' be groups and $\phi : G \to G'$ be a homomorphism. Prove: If G is cyclic and ϕ is onto, then G' is cyclic.

4. Find a subgroup of D_{2n} that is isomorphic to $\mathbb{Z}_n$. Prove it.

5. Let $\phi : G \to H$ be a homomorphism. Let x be in G.

   (a) Prove that the order of $\phi(x)$ divides the order of x.

   (b) If ϕ is an isomorphism, prove that the order of $\phi(x)$ equals the order of x. This shows that given a positive integer n, if G and H are isomorphic then they have the same number of elements of order n.