Math 5800
Homework # 3
Measure zero

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1. Let $A, B \subseteq \mathbb{R}$ with $A \subseteq B$.

   (a) Prove that if B has measure zero, then A has measure zero.

   (b) Prove that if A does not have measure zero, then B does not have measure zero.

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2. Given the set below, prove that S has measure zero, but do so directly from the definition of measure zero. That is, prove your answer without using any theorems or results from class or hw. You can only use the definition of measure zero.

   (a) $S = \{1, 2, 3, 4\}$

   (b) $S = \left\{ \frac{1}{n} \;\middle|\; n = 1, 2, 3, \ldots \right\}$

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3. Given the set S below, prove that either S has measure zero, or S does not have measure zero. In this problem you may use theorems from class to prove your result.

   (a) $S = \{1, \pi, 10\}$

   (b) $S = \mathbb{Q} \cap [0, 5)$

   (c) $S = [0, 1)$

   (d) $S = (a, b)$ where $a < b$

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4. Recall that $S \subseteq \mathbb{R}$ is called an almost everywhere set if $\mathbb{R} - S$ has measure zero.

   (a) Suppose that $S_1, S_2, \ldots S_n$ are almost everywhere sets. Prove that $\cap_{k=1}^{n} S_n$ is an almost everywhere set.
   Hint: Use DeMorgan's law: $\mathbb{R} - (\cap_{k=1}^{n} S_n) = \cup_{k=1}^{n} (\mathbb{R} - S_n)$

   (b) Suppose that $S_1, S_2, S_3, \ldots$ is a countably infinite number of almost everywhere sets. Prove that $\cap_{k=1}^{\infty} S_n$ is an almost everywhere set.
   Hint: Use DeMorgan's law: $\mathbb{R} - (\cap_{k=1}^{\infty} S_n) = \cup_{k=1}^{\infty} (\mathbb{R} - S_n)$

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5. For the following functions f and g, determine whether or not $f = g$ almost everywhere in $\mathbb{R}$.

   (a)

$$f(x) = 1 \text{ for all } x \in \mathbb{R} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ 1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

   (b)

$$f(x) = |x| \text{ for all } x \in \mathbb{R}$$

$$g(x) = \begin{cases} x & \text{if } x \geq 0 \text{ and } x \notin \{1, 6, 8\} \\ 5 & \text{if } x = 1 \\ 7 & \text{if } x = 6 \\ -1 & \text{if } x = 8 \\ -x & \text{if } x < 0 \text{ and } x \notin \mathbb{Z} \\ x & \text{if } x < 0 \text{ and } x \in \mathbb{Z} \end{cases}$$

   (c)

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-1, 2) \\ 0 & \text{if } x \notin (1, 2) \text{ and } x \neq 5 \\ -3 & \text{if } x = 5 \end{cases}$$

$$g(x) = \begin{cases} x^2 & \text{if } x \in [-1, 2] \\ 0 & \text{if } x \notin [-1, 2] \end{cases}$$

(d)

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ -1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

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6. (a) Let $A, B \subseteq \mathbb{R}$ with $A \subset B$. Prove that if A is an almost everywhere set, then B is an almost everywhere set.

   (b) Let $f, g, h : \mathbb{R} \to \mathbb{R}$. Suppose that $f = g$ almost everywhere in $\mathbb{R}$. Suppose that $h(x) = 5$ for almost all x in $\mathbb{R}$. Prove that $f(x) + h(x) = g(x) + 5$ for almost all x in $\mathbb{R}$.
