Math 5680

Homework # 3

Power series

1. Find the radius of convergence of the following power series.

   (a) $\sum_{n=1}^{\infty} n^2 z^n$

   (b) $\sum_{n=1}^{\infty} n! \frac{z^n}{n^n}$

   (c) $\sum_{n=0}^{\infty} \frac{z^{2n}}{4^n}$

   (d) $\sum_{n=0}^{\infty} \frac{(z-1)^n}{1+2^n}$

2. Compute the Taylor series for $f(z)$ centered at z_0. Determine the set that it converges on.

   (a) $f(z) = e^z$ centered at $z_0 = 1$.

   (b) $f(z) = 1/z$ centered at $z_0 = 1$.

   (c) $f(z) = e^{z^2}$ centered at $z_0 = 0$.

   (d) $f(z) = \sin(z^2)$ centered at $z_0 = 0$.

   (e) $f(z) = z^2 + z$ centered at $z_0 = 1$.

3. Compute the Taylor series for $f(z) = \dfrac{1}{(z-1)(z-2)}$ centered at $z_0 = 0$.

   Show that this series converges when $|z| < 1$.

4. Compute the first few terms of the Taylor series for $f(z)$ centered at the given point.

   (a) $f(z) = \sin(z)/z$ centered at $z_0 = 1$.

   (b) $f(z) = e^z \sin(z)$ centered at $z_0 = 0$.

5. Find power series for the following functions centered at $z_0 = 0$ and the radius of convergence of that series.

   (a) $\dfrac{1}{(1-z)^2}$

   (b) $\dfrac{1}{(1-z)^3}$

6. Let $f(z) = \sum a_n z^n$ have radius of convergence $R > 0$. Let

$$A = \{z \mid |z| < R\}.$$

   Let γ be a simple, piecewise smooth, closed curve lying inside of A. Prove that $\displaystyle\int_\gamma f = 0$.

7. (Isolation of zeroes of a non-constant analytic function) Suppose that f is analytic on an open set $A \subseteq \mathbb{C}$. Suppose that $z_0 \in A$ with $f(z_0) = 0$. Prove that either:

   (i) there is an $r > 0$ such that $D(z_0; r) \subseteq A$ and $f(z) = 0$ for all $z \in D(z_0; r)$,

   or

   (ii) there is an $r > 0$ such that $D(z_0; r) \subseteq A$ and $f(z) \neq 0$ for all $z \in D(z_0; r) - \{z_0\}$.

8. Let $f(z) = \sum a_n z^n$ have radius of convergence $R > 0$. Let

$$A = \{z \mid |z| < R\}.$$

   Let $z_0 \in A$. Let $\hat{R}$ be the radius of convergence for the Taylor series of f centered at z_0. Prove that

$$R - |z_0| \le \hat{R} \le R + |z_0|$$