Math 446 - Homework # 4

1. Are the following statements true or false?

   (a) $3 \equiv 5 (\mathrm{mod}\ 2)$

   (b) $11 \equiv -5 (\mathrm{mod}\ 5)$

   (c) $-31 \not\equiv 10 (\mathrm{mod}\ 3)$

   (d) $100 \equiv 12 (\mathrm{mod}\ 4)$

2. Prove the following: If x, y, z, a, b, n are integers with $n \geq 2$ then the following are true:

   (a) $x \equiv x (\mathrm{mod}\ n)$

   (b) If $x \equiv y (\mathrm{mod}\ n)$, then and $y \equiv x (\mathrm{mod}\ n)$.

   (c) If $x \equiv y (\mathrm{mod}\ n)$ and $y \equiv z (\mathrm{mod}\ n)$, then $x \equiv z (\mathrm{mod}\ n)$.

   (d) If $a \equiv b (\mathrm{mod}\ n)$ and $x \equiv y (\mathrm{mod}\ n)$, then $a + x \equiv b + y (\mathrm{mod}\ n)$.

   (e) If $a \equiv b (\mathrm{mod}\ n)$ and $x \equiv y (\mathrm{mod}\ n)$, then $ax \equiv by (\mathrm{mod}\ n)$.

   (f) We have that $x \equiv y (\mathrm{mod}\ n)$ if and only if $x = y + kn$ for some integer k.

3. In $\mathbb{Z}_4$, list ten elements from each of the following equivalence classes: $\overline{0}$, $\overline{-3}$, $\overline{2}$, $\overline{5}$.

4. Answer the following questions.

   (a) Is $\overline{0} = \overline{8}$ in $\mathbb{Z}_4$?

   (b) Is $\overline{-10} = \overline{-2}$ in $\mathbb{Z}_5$?

   (c) Is $\overline{1} = \overline{13}$ in $\mathbb{Z}_6$?

   (d) Is $\overline{2} = \overline{52}$ in $\mathbb{Z}_4$?

   (e) Is $\overline{-5} = \overline{19}$ in $\mathbb{Z}_4$?

5. Answer the following questions where the elements are from $\mathbb{Z}_8$.

   (a) Is $\overline{0} = \overline{12}$?

   (b) Is $\overline{-2} = \overline{14}$?

   (c) Is $\overline{-51} = \overline{-109}$?

(d) Is $\overline{3} = \overline{43}$?

6. Consider $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$. Calculate the following. For each answer $\overline{x}$ that you calculate, reduce it so that $0 \leq x \leq 6$.

   (a) $\overline{2} + \overline{6}$

   (b) $\overline{3} + \overline{4}$

   (c) $\overline{1473}$

   (d) $\overline{3} \cdot \overline{5}$

   (e) $\overline{2} \cdot \overline{3} + \overline{4} \cdot \overline{6}$

   (f) $\overline{5} \cdot \overline{2} + \overline{1} + \overline{2} \cdot \overline{4} \cdot \overline{6}$

7. Consider $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Calculate the following. For each answer $\overline{x}$ that you calculate, reduce it so that $0 \leq x \leq 3$.

   (a) $\overline{2} + \overline{3}$

   (b) $\overline{1} + \overline{3}$

   (c) $\overline{4630}$

   (d) $\overline{3} \cdot \overline{2}$

   (e) $\overline{2} \cdot \overline{2} + \overline{3} \cdot \overline{3}$

   (f) $\overline{3} \cdot \overline{2} + \overline{1} + \overline{2} + \overline{2} \cdot \overline{2} \cdot \overline{2}$

8. Suppose that x is an odd integer.

   (a) Prove that $\overline{x} = \overline{1}$ or $\overline{x} = \overline{3}$ in $\mathbb{Z}_4$.

   (b) Prove that $\overline{x}^2 = \overline{1}$ in $\mathbb{Z}_4$.

9. (a) Let p be a prime and x and y be integers. Suppose that $\overline{xy} = \overline{0}$ in $\mathbb{Z}_p$. Prove that either $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$.

   (b) Give an example where n is not prime with $\overline{xy} = \overline{0}$ but $\overline{x} \neq \overline{0}$ and $\overline{y} \neq \overline{0}$.

10. Let p be a prime. Suppose that $x^2 \equiv y^2 (\text{mod } p)$. Prove that either $p|(x+y)$ or $p|(x-y)$.

11. Let n be an integer with $n \geq 2$. Let $\overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_n$. Prove the following. (You will need to use the corresponding properties of the integers.)

(a) $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$.
(b) $\overline{a} + \overline{b} = \overline{b} + \overline{a}$.
(c) $\overline{a} \cdot (\overline{b} + \overline{c}) = \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c}$.
(d) $\overline{a} \cdot (\overline{b} \cdot \overline{c}) = (\overline{a} \cdot \overline{b}) \cdot \overline{c}$.
(e) $\overline{a} + (\overline{b} + \overline{c}) = (\overline{a} + \overline{b}) + \overline{c}$.

12. Prove that 4 does not divide $n^2 + 2$ for any integer n.

13. Prove that $15x^2 - 7y^2 = 1$ has no integer solutions.

14. Prove that $x^2 - 5y^2 = 2$ has no integer solutions.

15. Prove that the only integer solution to $x^2 + y^2 = 6z^2$ is $(x, y, z) = (0, 0, 0)$.

16. Let $n, x, y \in \mathbb{Z}$ with $n \geq 2$. Consider the elements $\overline{x}$ and $\overline{y}$ in $\mathbb{Z}_n$. Prove:

    (a) $\overline{x} = \overline{y}$ if and only if $x \equiv y (\text{mod } n)$.
    (b) Either $\overline{x} \cap \overline{y} = \emptyset$ or $\overline{x} = \overline{y}$.

17. Prove that if a positive integer $x > 1$ ends in a 7 then it is not a square. For example, $x = 137$ is not a square.