Math 4680 - Homework # 4

Limits and Continuity

1. Let $a, b, c, z_0 \in \mathbb{C}$ be constants. Show the following.
   (a) $\lim_{z \to z_0} c = c$
   (b) $\lim_{z \to z_0} az + b = az_0 + b$
   (c) $\lim_{z \to z_0} z^2 + c = z_0^2 + c$
   (d) $\lim_{z \to z_0} \text{Re}(z) = \text{Re}(z_0)$

2. Show that $f(z) = \overline{z}$ is continuous for all $z_0 \in \mathbb{C}$.

3. Show that $f(z) = |z|$ is continuous for all $z_0 \in \mathbb{C}$

4. Let $f : A \to \mathbb{C}$. Let $z_0 \in \mathbb{C}$ where some deleted neighborhood of z_0 sits inside of A. Suppose that the limit of $f(z)$ at z_0 exits. Prove that the limit is unique. [That is, prove that if $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} f(z) = L_2$, then $L_1 = L_2$.]

5. Let $f : A \to \mathbb{C}$ be continuous on an open set A. Let $z_0 \in A$ with $f(z_0) \neq 0$, Prove that there is a neighborhood $D(z_0; r) \subseteq A$ such that $f(z) \neq 0$ for all $z \in D(z_0; r)$.

6. Let $f : \mathbb{C} \to \mathbb{C}$. Prove that f is continuous on all of $\mathbb{C}$ if and only if $f^{-1}(A)$ is open for every open set $A \subseteq \mathbb{C}$.