Math 4570 - Homework # 5

Eigenvalues, Eigenvectors, and Diagonalization

1. For each linear transformation T: (i) find all the eigenvalues of T, (ii) find a basis for each eigenspace of T, (iii) state the algebraic and geometric multiplicities of each eigenvalue of T, and verify that the geometric multiplicity is bounded by the algebraic multiplicity, (iv) determine if T is diagonalizable.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4a + c \\ 2a + 3b + 2c \\ a + 4c \end{pmatrix}$

   (b) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(f(x)) = f(x) + (x + 1)f'(x)$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + b \\ 3b \\ 4c \end{pmatrix}$

   (d) $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by $T\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z + iw \\ iz + w \end{pmatrix}$

   (e) $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(f(x)) = f'(x) + f''(x)$

2. Let V be a finite-dimensional vector space over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V. Let $I_V : V \to V$ be the identity transformation given by $I_V(x) = x$ for all $x \in V$. Show that $[I_V]_\beta = I$ where I is the $n \times n$ identity matrix.

3. Let F be a field. A matrix $A \in M_{n,n}(F)$ is said to be diagonalizable iff there exists an invertible matrix $Q \in M_{n,n}(F)$ such that $Q^{-1}AQ = D$ where D is a diagonal matrix.

   (a) Let V be a finite-dimensional vector space over a field F. Let β be an ordered basis for V and let $T : V \to V$ be a linear transformation. Then T is diagonalizable if and only if $[T]_\beta$ is diagonalizable.

   (b) Let A be an $n \times n$ matrix over a field F. Then A is diagonalizable if and only if L_A is diagonalizable.

4. Let V be a finite-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation. Let β and γ be ordered bases for V. Then $\det([T]_\beta) = \det([T]_\gamma)$.

5. Let V be a finite-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation. Recall that $\det(T)$ is defined to be $\det([T]_\beta)$ where β is any basis of V.

   (a) T is invertible if and only if $\det(T) \neq 0$.

   (b) If T is invertible, then $\det(T^{-1}) = (\det(T))^{-1}$.

   (c) If $S : V \to V$ is another linear transformation, then $\det(T \circ S) = \det(T)\det(S)$.

6. Let V be a finite-dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Let λ be an eigenvalue of T. Recall that

$$E_\lambda(T) = \{x \in V | T(x) = \lambda x\}$$

   (a) $E_\lambda(T)$ is a subspace of V.

   (b) If $v_1, v_2, \ldots, v_r \in E_\lambda(T)$, then $c_1v_1 + c_2v_2 + \cdots + c_rv_r \in E_\lambda(T)$ for any scalars $c_1, c_2, \ldots, c_r \in F$.