Homework 6 Solutions

① Let $\{\mathcal{O}_\alpha\}$ be an open cover of $S = \{x_1, x_2, \ldots, x_n\}$.

By definition of open cover, for each i there exists $\mathcal{O}_{\alpha_i}$ such that $x_i \in \mathcal{O}_{\alpha_i}$.

Then, $\{\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}, \ldots, \mathcal{O}_{\alpha_n}\}$ is a finite subcover of S.

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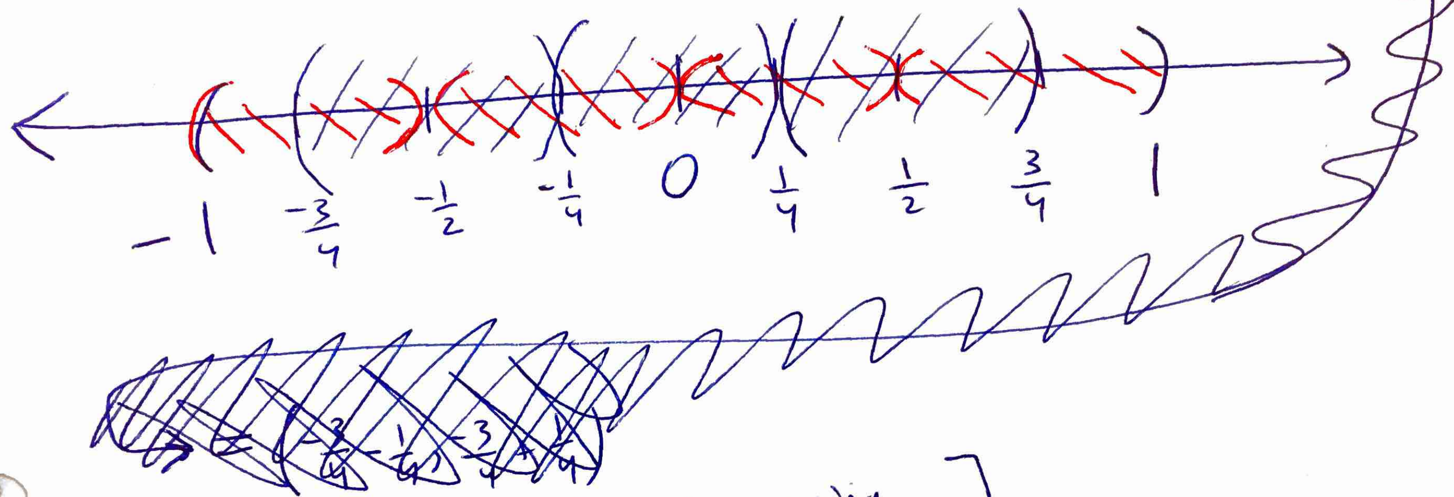
② ~~Suppose that~~ Let $\mathcal{O}_n = (n-1, n+1)$.

Suppose that $\{\mathcal{O}_{n_1}, \mathcal{O}_{n_2}, \ldots, \mathcal{O}_{n_k}\}$ is a finite ~~[illegible]~~ subset of X, where $n_1 < n_2 < \ldots < n_k$.

Then, $\bigcup_{i=1}^{k} \mathcal{O}_{n_i} \neq [1, \infty)$ since $n_k + 2 \in [1, \infty)$ but $n_k + 2 \notin \bigcup_{i=1}^{k} \mathcal{O}_{n_i}$.

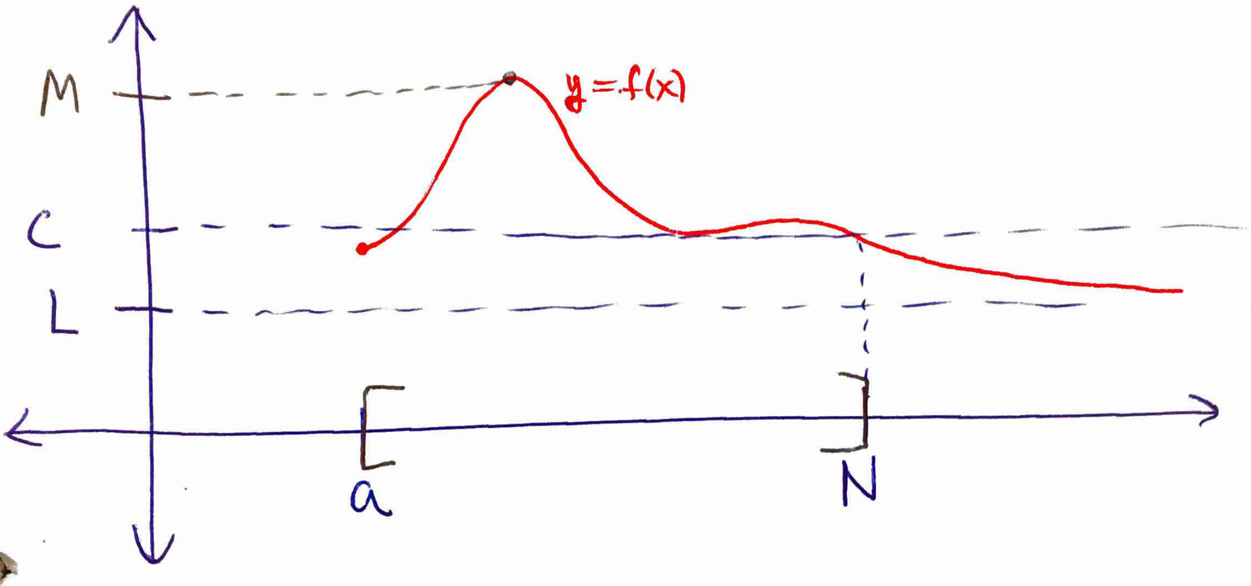
3

$$(-1,1) = \left(-1,-\tfrac{1}{2}\right) \cup \left(-\tfrac{3}{4},-\tfrac{1}{4}\right) \cup \left(-\tfrac{1}{2},0\right) \cup \left(-\tfrac{1}{4},\tfrac{1}{4}\right) \cup \left(0,\tfrac{1}{2}\right) \cup \left(\tfrac{1}{4},\tfrac{3}{4}\right) \cup \left(\tfrac{1}{2},1\right) =$$



[I got these intervals by using $x = -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$]

4. By HW #3, problem 3, there exists $C>0$ and $N>0$ such that if $x \geq N$ then $|f(x)| < C$.



Since $[a,N]$ is closed and bounded, by class theorem, f is bounded on $[a,b]$ because it's continuous. Hence there exists $M>0$ such that $|f(x)| < M$ for all $x \in [a,N]$.

Then, $|f(x)| < \max\{M, C\}$ for all $x \geq a$.

out of order

(5)

(b) Since A is compact, A is closed and bounded. Thus, there exists M_A such that $|a| \le M_A$ for all $a \in A$.
Since B is compact, B is closed and bounded. Thus, there exists M_B such that $|b| \le M_B$ for all $b \in B$.
Hence $|x| \le \max\{M_A, M_B\}$ for all $x \in A \cup B$.
So, $A \cup B$ is bounded.
Also, $A \cup B$ is closed by hw #5.

(a) Since A is compact, A is closed and bounded. Since $A \cap B \subseteq A$, $A \cap B$ is also bounded. By hw #5, since A and B are both closed, $A \cap B$ is closed.

(c) Consider the sets $A_n = [n, n+1]$.
Each A_n is compact.
However, $[1,\infty) = \bigcup_{n=1}^{\infty} A_n$ is not compact since $\mathbb{R}$ is not bounded.

(d) Let A_n be compact for $n \geq 1$.
We will show that $\bigcap_{n=1}^{\infty} A_n$ is compact by showing that $\bigcap_{n=1}^{\infty} A_n$ is closed and bounded.

Bounded

Since A_1 is bounded, there exists $M>0$ where $|x| \leq M$ for all $x \in A_1$. Since $\bigcap_{n=1}^{\infty} A_n \subseteq A_1$ we know that $|x| \leq M$ for all $x \in \bigcap_{n=1}^{\infty} A_n$.
So $\bigcap_{n=1}^{\infty} A_n$ is bounded.

closed

Consider $\mathbb{R} \setminus \left[\bigcap_{n=1}^{\infty} A_n\right] = \bigcup_{n=1}^{\infty} [\mathbb{R} \setminus A_n]$. We will show that $\mathbb{R} \setminus \left[\bigcap_{n=1}^{\infty} A_n\right]$ is open and hence $\bigcap_{n=1}^{\infty} A_n$ is closed.
Since each A_n is closed, we know that $\mathbb{R} \setminus A_n$ is open for all n.
We will be done if we prove this result:

Lemma: If B_n is open for $n \geq 1$, then $\bigcup_{n=1}^{\infty} B_n$ is open.

Pf of lemma: Let $x \in \bigcup_{n=1}^{\infty} B_n$. Then $x \in B_m$ for some m. Since B_m is open, there exists $a, b \in \mathbb{R}$ with $x \in (a,b) \subseteq B_m$. So, $x \in (a,b) \subseteq B_m \subseteq \bigcup_{n=1}^{\infty} B_n$. So x is an interior point of $\bigcup_{n=1}^{\infty} B_n$. $\boxed{\text{Lemma}}$

So, $\mathbb{R} \setminus \left[\bigcap_{n=1}^{\infty} A_n\right]$ is open. Thus, ~~[scribbled out]~~ $\bigcap_{n=1}^{\infty} A_n$ is closed. $\blacksquare$

~~(d) Let $C_n = [\frac{1}{n}, 1]$. Then C_n is compact for each $n \in \mathbb{N}$ with $n \geq 1$. And $(0,1] = \bigcup_{n=1}^{\infty} C_n$. But $(0,1]$ is not compact since it is not closed because 0 is a limit point of $(0,1]$ that is not contained in $(0,1]$. [closed sets contain all their limit points]~~

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⑥ (a) $(0,1]$ is not closed since 0 is a limit point of $(0,1]$ that is not contained in the set $(0,1]$.
So, $(0,1]$ is not compact.

(b) S is bounded, since $-10 \leq x \leq 2$ for all $x \in S$. And S is closed since it is the union of two closed sets. ← (hw #5 result)

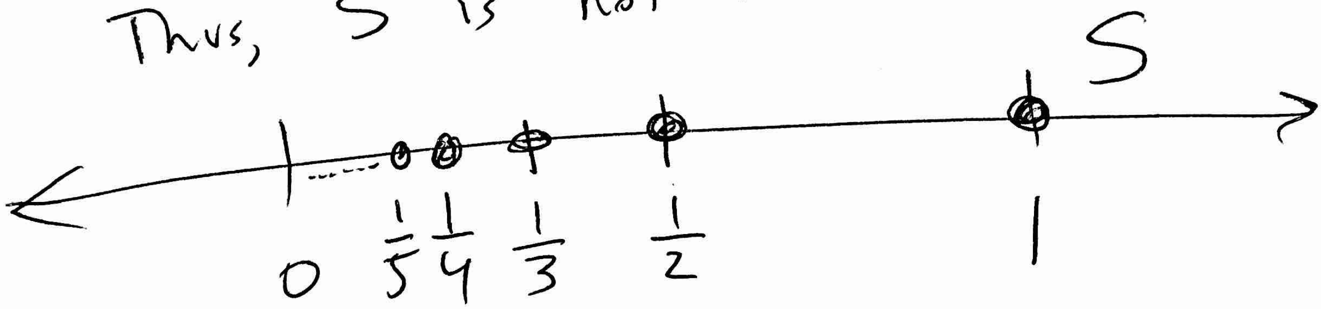
So, S is compact since it is closed and bounded.

(c) S is not bounded and thus is not compact.

(d) S is not bounded and thus is not compact.

(e) S is bounded since $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$.

However, 0 is a limit point of S that is not contained in S. Thus, S is not closed.



Note: $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ would be compact. It's called the "closure" of $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

⑦ Suppose $\lim_{n\to\infty} a_n = L$ and $A = \{a_n \mid n \in \mathbb{N}\} \cup \{L\}$.
We show that A is compact by showing that A is closed and bounded.

<u>bounded:</u>

Since $\lim_{n\to\infty} a_n = L$, from a theorem in class,
$\exists M > 0$ where $|a_n| \leq M$ for all n.
Let $\hat{M} = \max\{M, |L|\}$.
Then, $|x| \leq \hat{M}$ for all $x \in A$.
Thus, A is bounded.

<u>closed:</u>

We show that A is closed by showing that $\mathbb{R} \setminus A$ is open.
Let $x \in \mathbb{R} \setminus A$.
Since $x \notin A$, we know that $x \neq L$.
Let $\varepsilon = \frac{|L - x|}{2} > 0$.
Since $\lim_{n\to\infty} a_n = L$, there exists an integer $N > 0$ where
if $n \geq N$ then $|a_n - L| < \varepsilon$.
Since $x \notin A$, $x \neq a_i$ for all i.
So, $|x - a_i| > 0$, for all i.

Let $\delta = \min\{\varepsilon, |x-a_1|, |x-a_2|, \ldots, |x-a_{N-1}|\}$

Then $\delta > 0$ and there are no points of A inside of $(x-\delta, x+\delta)$.

So, $(x-\delta, x+\delta) \subseteq \mathbb{R} \setminus A$.

Thus, $\mathbb{R} \setminus A$ is open. $\blacksquare$

PICTURE FOR $N=5$