Homework 7 Solutions

①

(a) $f(x) = \frac{1}{x}$ is uniformly continuous on $[1,\infty)$

pf: Let $\varepsilon > 0$.

If $x,y \in [1,\infty)$, then

$$\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{|x||y|} = \frac{|x-y|}{|x||y|} \leq |x-y|$$

since $x,y \geq 1$ we have $\frac{1}{|x|} \leq 1$ and $\frac{1}{|y|} \leq 1$

Let $\delta = \varepsilon$. Then if $x,y \in [1,\infty)$ and $|x-y| < \delta$ then

$$\left|\frac{1}{x}-\frac{1}{y}\right| \leq |x-y| < \delta = \varepsilon. \quad \square$$

Since if say $\delta \geq 1$ is arbitrary and $\delta' < 1 \leq \delta$ then if $|x-y| < \delta'$, then $|x-y| < \delta$

(b) $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

Let $\varepsilon = 1$.

Suppose $\delta > 0$. We may assume that $\delta < 1$.

Let $x \in (0,1)$ with $x < \delta$.

Set $y = \frac{x}{2}$. Then $y \in (0,1)$.

$0 < x < 1$

So, $x,y \in (0,1)$ and $|x-y| = \left|x - \frac{x}{2}\right| = \left|\frac{x}{2}\right| = \frac{x}{2} < \frac{\delta}{2} < \delta$.

And
$$\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{x} - \frac{1}{\left(\frac{x}{2}\right)}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \left|-\frac{1}{x}\right|$$
$$= \left|\frac{1}{x}\right| = \frac{1}{x} > \frac{1}{\delta} > 1 = \varepsilon.$$

That is, there exist $x,y \in (0,1)$ with $|x-y| < \delta$ but $\left|\frac{1}{x}-\frac{1}{y}\right| > \varepsilon$. Since δ was arbitrary, we are done. $\square$

(c)

We need a lemma.

Lemma: If $r \in \mathbb{R}$, then $\left|\frac{r}{1+r^2}\right| \le 1$.

pf: If $r \le 1$, then $\left|\frac{r}{1+r^2}\right| = \frac{|r|}{|1+r^2|} \le \frac{1}{|1+r^2|} = \frac{1}{1+r^2} \le \frac{1}{1} = 1$

$\boxed{1+r^2 \ge 1}$

If $r > 1$, then $\left|\frac{r}{1+r^2}\right| = \frac{|r|}{1+r^2} \le \frac{|r|}{r^2} = \frac{|r|}{|r^2|} = \left|\frac{r}{r^2}\right|$

$= \left|\frac{1}{r}\right| = \frac{1}{r} < 1$

$\boxed{r > 1}$

$\boxed{\text{Lemma}}$

Now we prove (c).

Let $\varepsilon > 0$.

Let $x, y \in \mathbb{R}$.

Then

$$\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \left|\frac{(1+y^2)-(1+x^2)}{(1+x^2)(1+y^2)}\right| = \left|\frac{y^2 - x^2}{(1+x^2)(1+y^2)}\right| = \frac{|y^2 - x^2|}{|1+x^2||1+y^2|}$$

$$= \frac{|x^2 - y^2|}{|1+x^2||1+y^2|} = \frac{|x+y||x-y|}{|1+x^2||1+y^2|} = \frac{|x+y|}{|1+x^2||1+y^2|} \cdot |x-y|$$

$$\le \left[\frac{|x|+|y|}{|1+x^2||1+y^2|}\right]|x-y| = \left(\frac{|x|}{|1+x^2||1+y^2|} + \frac{|y|}{|1+x^2||1+y^2|}\right)|x-y|$$

$$\le \left[\frac{|x|}{|1+x^2|} + \frac{|y|}{|1+y^2|}\right]|x-y| \le [1+1]|x-y| = 2|x-y|$$

$\boxed{\begin{array}{c}|1+x^2| \ge 1 \\ |1+y^2| \ge 1\end{array}}$

$\boxed{\text{lemma}}$

Set $\delta = \frac{\varepsilon}{2}$. If $x, y \in \mathbb{R}$ and $|x-y| < \delta$, then

$$\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| \le 2|x-y| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

~~Homework 4 Solutions~~

~~1) Let $\varepsilon > 0$.~~

~~Note that if $x, y \in [1, \infty)$ then~~

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{|x||y|} = \frac{|x-y|}{|x||y|} \leq |x-y|.$$

~~Since $|x| \geq 1$ and $|y| \geq 1$ we have $\frac{1}{|x|} \leq 1$ and $\frac{1}{|y|} \leq 1$~~

~~Let $\delta = \varepsilon$. Then if $x, y \in [1, \infty)$ and $|x-y| < \delta$ then~~

$$\left|\frac{1}{x} - \frac{1}{y}\right| \leq |x-y| < \delta = \varepsilon.$$

~~from above~~

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② Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both uniformly continuous on D.

Let $\varepsilon > 0$.

~~Let $x, y \in D$~~

Since f is uniformly continuous on D there exists $\delta_1 > 0$ where if $x, y \in D$ and $|x-y| < \delta_1$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$.

Since g is uniformly continuous on D there exists $\delta_2 > 0$ where if $x, y \in D$ and $|x-y| < \delta_2$ then $|g(x) - g(y)| < \varepsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$.
Then if $x, y \in D$ and $|x-y| < \delta$
then

$$|(f+g)(x) - (f+g)(y)| = |f(x) + g(x) - f(y) - g(y)|$$
$$= |f(x) - f(y) + g(x) - g(y)|$$
$$\leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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3. Let $\varepsilon > 0$.
Since f is bounded on D, there exists $M_1 > 0$
so that $|f(d)| \leq M_1$ for all $d \in D$.
Since g is bounded on D, there exists $M_2 > 0$
so that $|g(d)| \leq M_2$ for all $d \in D$.
Since f is uniformly continuous on D there exists
$\delta_1 > 0$ so that if $x, y \in D$ and $|x-y| < \delta_1$
then $|f(x) - f(y)| < \frac{\varepsilon}{2M_2}$.
Since g is uniformly continuous on D there exists
$\delta_2 > 0$ so that if $x, y \in D$ and $|x-y| < \delta_2$
then $|g(x) - g(y)| < \frac{\varepsilon}{2M_1}$.
Let $\delta = \min\{\delta_1, \delta_2\}$.

If $x, y \in D$ and $|x-y| < \delta$ then

$$|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)|$$

$$\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)|$$

$$= |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)|$$

$$< M_2 \cdot \frac{\varepsilon}{2M_2} + M_1 \cdot \frac{\varepsilon}{2M_1} = \varepsilon.$$

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(4) Let $\varepsilon > 0$.

Since f is uniformly continuous on $\mathbb{R}$ there exists $\delta' > 0$ so that if $a, b \in \mathbb{R}$ and $|a-b| < \delta'$ then $|f(a) - f(b)| < \varepsilon$.

Since g is uniformly continuous there exists $\delta > 0$ so that if $x, y \in \mathbb{R}$ and $|x-y| < \delta$ then $|g(x) - g(y)| < \delta'$.

Therefore if $x, y \in \mathbb{R}$ and $|x-y| < \delta$ then $|g(x) - g(y)| < \delta'$ and hence $|f(g(x)) - f(g(y))| < \varepsilon$.

(5) Let $\varepsilon > 0$.
Since f is uniformly continuous on D there exists $\delta > 0$ so that if $a, b \in D$ and $|a-b| < \delta$ then $|f(a) - f(b)| < \varepsilon$.
Since x_n is Cauchy, there exists $N > 0$ so that if $n, m \geq N$ then $|x_n - x_m| < \delta$.
Therefore if $n, m \geq N$ then $|x_n - x_m| < \delta$ and hence $|f(x_n) - f(x_m)| < \varepsilon$.