Math 4300 - Homework # 7

The plane separation axiom and convex sets

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1. In the Euclidean plane, let $A = (1,4)$, $B = (-1,1)$. Let $\ell = \overleftrightarrow{AB}$.

   (a) Draw a picture of the two half planes that are determined by ℓ.

   (b) Let $P = (-2,1)$ and $Q = (0,0)$. Determine if P and Q on the same side of ℓ or opposite sides of ℓ.

   (c) Let $P = (3,1)$ and $Q = (0,0)$. Determine if P and Q on the same side of ℓ or opposite sides of ℓ.

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2. In the Hyperbolic plane, let $A = (1,2)$, $B = (3,4)$. Let $\ell = \overleftrightarrow{AB}$.

   (a) Draw a picture of the two half planes that are determined by ℓ.

   (b) Let $P = (-2,1)$ and $Q = (5,1)$. Determine if P and Q on the same side of ℓ or opposite sides of ℓ.

   (c) Let $P = (-2,1)$ and $Q = (10,2)$. Determine if P and Q on the same side of ℓ or opposite sides of ℓ.

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3. Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry.

   (a) Let X and Y be distinct points from $\mathscr{P}$. Let $\ell = \overleftrightarrow{XY}$. Let $f : \ell \to \mathbb{R}$ be a ruler for ℓ. Prove: If $f(X) < f(Y)$ then

   $$\overline{XY} = \{D \in \mathscr{P} \mid f(X) \leq f(D) \leq f(Y)\}$$

   Prove that if If $f(Y) < f(X)$ then

   $$\overline{XY} = \{D \in \mathscr{P} \mid f(Y) \leq f(D) \leq f(X)\}$$

(b) Let X and Y be distinct points from $\mathscr{P}$. Let $\ell = \overleftrightarrow{XY}$. Let $f : \ell \to \mathbb{R}$ be a ruler for ℓ. Prove: If $f(X) < f(Y)$ then

$$\overrightarrow{XY} = \{C \in \mathscr{P} \mid f(X) \leq f(C)\}$$

Prove that if If $f(Y) < f(X)$ then

$$\overrightarrow{XY} = \{C \in \mathscr{P} \mid f(C) \leq f(X)\}$$

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4. Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let $S, T \subseteq \mathscr{P}$ be convex sets. Prove that $S \cap T$ is convex.

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5. Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry. Let A and B be distinct points. Prove that the following sets are convex.

   (a) $\emptyset$, which is the empty set

   (b) $\{A\}$, a set with just one point

   (c) The set $\mathscr{P}$ of all points in the geometric space

   (d) $\overline{AB}$

   (e) $\text{int}(\overline{AB})$ where $\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$

   (f) $\overleftrightarrow{AB}$

   (g) $\overrightarrow{AB}$

   (h) $\text{int}(\overrightarrow{AB})$ where $\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}$

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6. Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry satisfying the PSA. Let ℓ be a line from $\mathscr{L}$. Let P, Q be points in $\mathscr{P}$ where $P \notin \ell$ and $Q \notin \ell$. We have that:

   (a) P and Q are on opposite sides of ℓ if and only if $\overline{PQ} \cap \ell \neq \emptyset$.

   (b) P and Q are on the same side of ℓ if and only if $\overline{PQ} \cap \ell = \emptyset$.

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7. Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry satisfying the PSA. Let P, Q, R be points in $\mathscr{P}$ and let ℓ be a line from $\mathscr{L}$. If P and Q are on opposite sides of ℓ, and Q and R are on opposite sides of ℓ, then P and R are on the same side of ℓ.

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8. Let $(\mathscr{P}, \mathscr{L}, d)$ be a metric geometry satisfying the PSA. Let P, Q, R be points in $\mathscr{P}$ and let ℓ be a line from $\mathscr{L}$. If P and Q are on opposite sides of ℓ, and Q and R are on the same side of ℓ, then P and R are on opposite sides of ℓ.
