$\boxed{\text{Hw \#9}}$

(1) $\mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}\}$

$H = \{\bar{0}, \bar{4}, \bar{8}\}$

left cosets:

$$\left.\begin{aligned} H &= \{\bar{0}, \bar{4}, \bar{8}\} \\ \bar{1} + H &= \{\bar{1}, \bar{5}, \bar{9}\} \\ \bar{2} + H &= \{\bar{2}, \bar{6}, \overline{10}\} \\ \bar{3} + H &= \{\bar{3}, \bar{7}, \overline{11}\} \end{aligned}\right\} \quad G/H = \{H, \bar{1}+H, \bar{2}+H, \bar{3}+H\}$$

$$\left.\begin{aligned} &\bar{5}+H \neq H \\ &(\bar{5}+H)+(\bar{5}+H) = \overline{10}+H = \bar{2}+H \neq \bar{0}+H \\ &(\bar{5}+H)+(\bar{5}+H)+(\bar{5}+H) = \overline{15}+H = \bar{3}+H \neq \bar{0}+H \\ &(\bar{5}+H)+(\bar{5}+H)+(\bar{5}+H)+(\bar{5}+H) = \overline{20}+H = \bar{8}+H = \bar{0}+H \end{aligned}\right\} \begin{matrix} \bar{5}+H \\ \text{has} \\ \text{order} \\ 4 \end{matrix}$$

$$\left.\begin{aligned} &\bar{6}+H = \bar{2}+H \neq H \\ &(\bar{6}+H)+(\bar{6}+H) = \bar{0}+H \end{aligned}\right\} \quad \bar{6}+H \text{ has order } 2$$

② left cosets:

$$H = \{ (\bar{0},\bar{0}), (\bar{1},\bar{1}), (\bar{2},\bar{2}), (\bar{3},\bar{3}) \}$$

$$(\bar{1},\bar{0}) + H = \{ (\bar{1},\bar{0}), (\bar{2},\bar{1}), (\bar{3},\bar{2}), (\bar{0},\bar{3}) \}$$

$$(\bar{2},\bar{0}) + H = \{ (\bar{2},\bar{0}), (\bar{3},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{3}) \}$$

$$(\bar{3},\bar{0}) + H = \{ (\bar{3},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{2}), (\bar{2},\bar{3}) \}$$

$$G/H = \{ H, (\bar{1},\bar{0})+H, (\bar{2},\bar{0})+H, (\bar{3},\bar{0})+H \}$$

$$(\bar{3},\bar{1}) + H \neq H$$

$$\big((\bar{3},\bar{1}) + H\big) + \big((\bar{3},\bar{1}) + H\big) = (\bar{6},\bar{2}) + H = (\bar{2},\bar{2}) + H = (\bar{0},\bar{0}) + H$$

So, $(\bar{3},\bar{1})+H$ has order 2

$$(\bar{2},\bar{3}) + H = (\bar{3},\bar{0}) + H \neq (\bar{0},\bar{0}) + H$$

$$[(\bar{2},\bar{3}) + H] + [(\bar{2},\bar{3}) + H] = (\bar{4},\bar{6}) + H = (\bar{0},\bar{2}) + H \neq (\bar{0},\bar{0}) + H$$

$$[(\bar{2},\bar{3}) + H] + [(\bar{2},\bar{3}) + H] + [(\bar{2},\bar{3}) + H] = (\bar{2},\bar{1}) + H \neq (\bar{0},\bar{0}) + H$$

$$[(\bar{2},\bar{3}) + H] + [(\bar{2},\bar{3}) + H] + [(\bar{2},\bar{3}) + H] + [(\bar{2},\bar{3}) + H] = (\bar{0},\bar{0}) + H$$

So $(\bar{2},\bar{3})+H$ has order 4

(3) $\mathbb{Z}_2 \times \mathbb{Z}_4 / H \cong \mathbb{Z}_2$

proof:

Let $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ be $\varphi(\bar{x}, \bar{y}) = \bar{x}$.

Then $\ker(\varphi) = \{ (\bar{x}, \bar{y}) \mid \varphi(\bar{x}, \bar{y}) = \bar{0} \}$

$= \{ (\bar{0}, \bar{y}) \mid \bar{y} \in \mathbb{Z}_4 \}$

$= \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}) \} = H$.

Note that φ is onto $\mathbb{Z}_2$ since $\varphi(\bar{0},\bar{0}) = \bar{0}$ and $\varphi(\bar{1},\bar{0}) = \bar{1}$. Thus, by the first isomorphism theorem

$$\mathbb{Z}_2 \times \mathbb{Z}_4 / H = \mathbb{Z}_2 \times \mathbb{Z}_4 / \ker(\varphi) \cong \varphi(\mathbb{Z}_2 \times \mathbb{Z}_4) = \mathbb{Z}_2.$$

④ $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ $\quad H = \{(\bar{0},\bar{0}), (\bar{0},\bar{2})\}$

left cosets:

$(\bar{0},\bar{0}) + H = \{(\bar{0},\bar{0}), (\bar{0},\bar{2})\}$
$(\bar{1},\bar{0}) + H = \{(\bar{1},\bar{0}), (\bar{1},\bar{2})\}$
$(\bar{0},\bar{1}) + H = \{(\bar{0},\bar{1}), (\bar{0},\bar{3})\}$
$(\bar{1},\bar{1}) + H = \{(\bar{1},\bar{1}), (\bar{1},\bar{2})\}$

So, $G/H = \{(\bar{0},\bar{0})+H, (\bar{1},\bar{0})+H, (\bar{0},\bar{1})+H, (\bar{1},\bar{1})+H\}$

Note that $(\bar{1},\bar{0})+H$, $(\bar{0},\bar{1})+H$, and $(\bar{1},\bar{1})+H$ all have order 2. Hence G/H is not cyclic. So, $G/H \not\cong \mathbb{Z}_4$. Thus, by another hw assignment $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(5) $G = \mathbb{Z} \times \mathbb{Z}$

$$H = \langle (1,1) \rangle = \{ \dots, (-2,-2), (-1,-1), (0,0), (1,1), (2,2), \dots \}$$

We show that $G/H \cong \mathbb{Z}$.

Define $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $\varphi(n,m) = n - m$.

Then φ is a homomorphism since

$$\varphi((a,b)+(x,y)) = \varphi(a+x, b+y) = a+x-(b+y) = (a-b)+(x-y)$$
$$= \varphi(a,b) + \varphi(x,y).$$

Also, $\ker(\varphi) = \{ (n,m) \mid \underset{n,m \in \mathbb{Z}}{\varphi(n,m) = 0} \} = \{ (n,m) \mid \underset{n,m \in \mathbb{Z}}{n - m = 0} \}$
$$= \{ (n,m) \mid \underset{n,m \in \mathbb{Z}}{n = m} \} = \{ (a,a) \mid a \in \mathbb{Z} \} = \langle (1,1) \rangle.$$

Note that $\varphi(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}$. Why? Let $x \in \mathbb{Z}$, then $(x,0) \in \mathbb{Z} \times \mathbb{Z}$ and $\varphi(x,0) = x - 0 = x$.

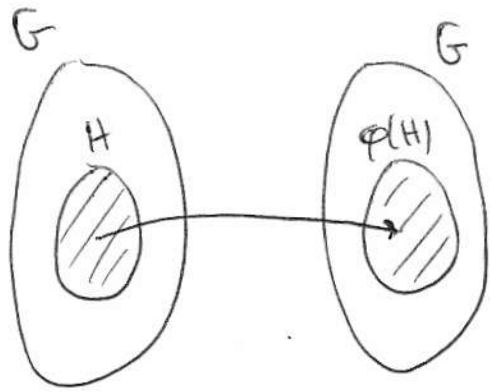
Hence,

$$G/H = G/\ker(\varphi) \cong \varphi(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}.$$

⑥ Let $g \in G$. We prove that $gHg^{-1} = H$ which shows that H is normal in G.

Define $\varphi : G \to G$ by $\varphi(x) = gxg^{-1}$.

By a previous hw assignment φ is an isomomorphism. Thus, $H \cong \varphi(H)$.



Since φ is 1-1 and onto, $|\varphi(H)| = |H|$. Since φ is a homomorphism, $\varphi(H) \leq G$. Thus, by hypothesis, $H = \varphi(H)$. So, $H = gHg^{-1}$.

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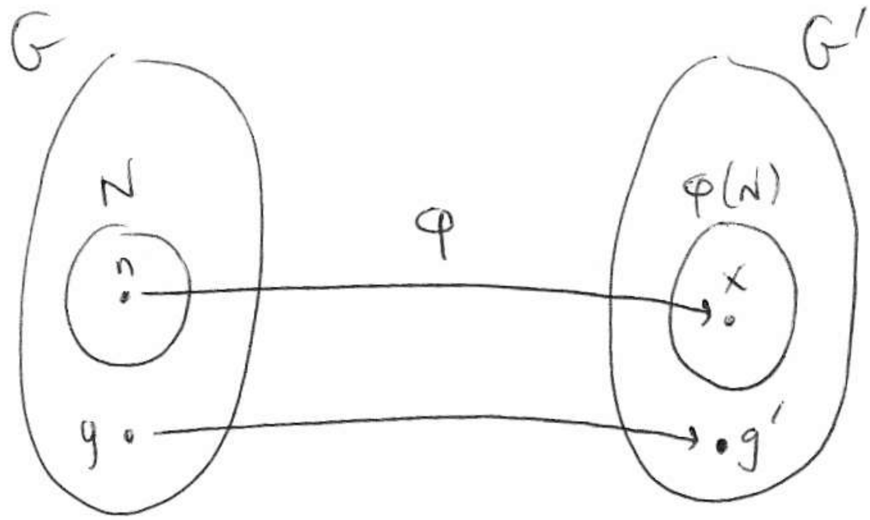
⑦ By a previous hw assignment $H \cap K$ is a subgroup of G. So we just need to show that $H \cap K$ is normal in G. Let $x \in H \cap K$ and $g \in G$. Then, $x \in H$ and $x \in K$. Since $H \trianglelefteq G$, $gxg^{-1} \in H$. Since $K \trianglelefteq G$, $gxg^{-1} \in K$. So, $gxg^{-1} \in H \cap K$. Hence $H \cap K \trianglelefteq G$.

(8) In class we showed that $\varphi(N)$ is a subgroup of G'. Let's show that $\varphi(N)$ is normal in G'. Let $x \in \varphi(N)$ and $g' \in G'$. Since φ is onto there exists $g \in G$ with $\varphi(g) = g'$. Since $x \in \varphi(N)$ there exists $n \in N$ with $\varphi(n) = x$.

Then $g' x (g')^{-1} = \varphi(g)\varphi(n)\varphi(g^{-1})$ $= \varphi(gng^{-1})$. Since $N \trianglelefteq G$, $gng^{-1} \in N$. So, $\varphi(gng^{-1}) \in \varphi(N)$. Thus, $\varphi(N) \trianglelefteq G'$.



(9) From class we know that $\varphi^{-1}(N') \leq G$. Let's show that it's normal. Let $x \in \varphi^{-1}(N')$ and $g \in G$. Since $x \in \varphi^{-1}(N')$ we know that $\varphi(x) \in N'$. Note that $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} \in N'$ since $N' \trianglelefteq G'$. Hence $gxg^{-1} \in \varphi^{-1}(N')$. So, $\varphi^{-1}(N') \trianglelefteq G$.