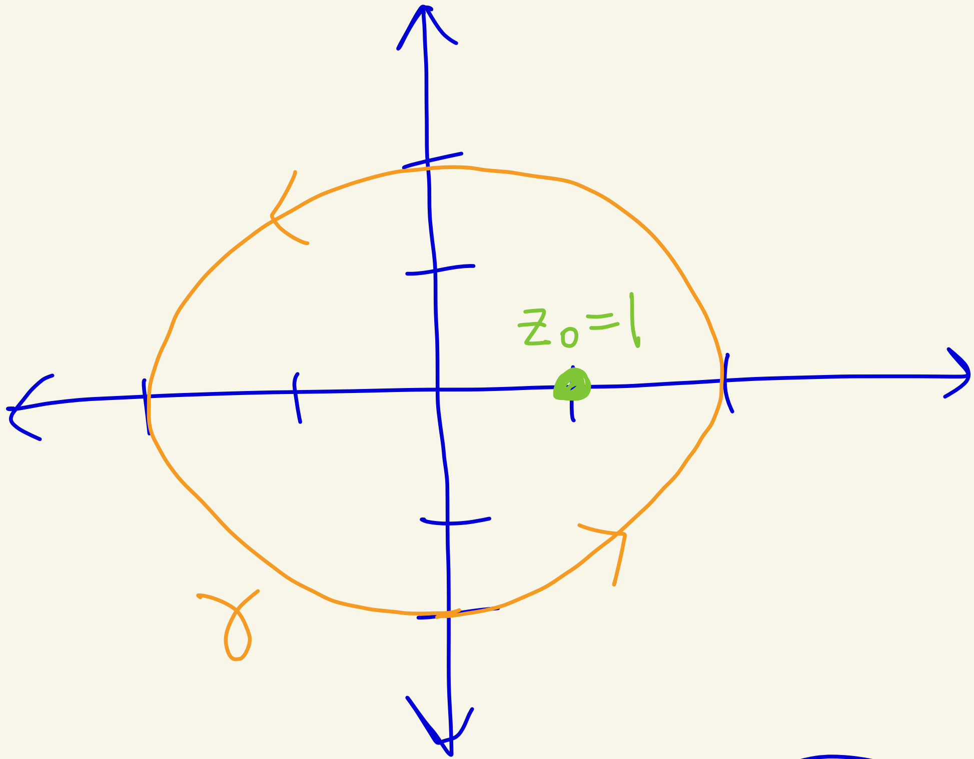
4680 - HW 10

Solutions

① (a)



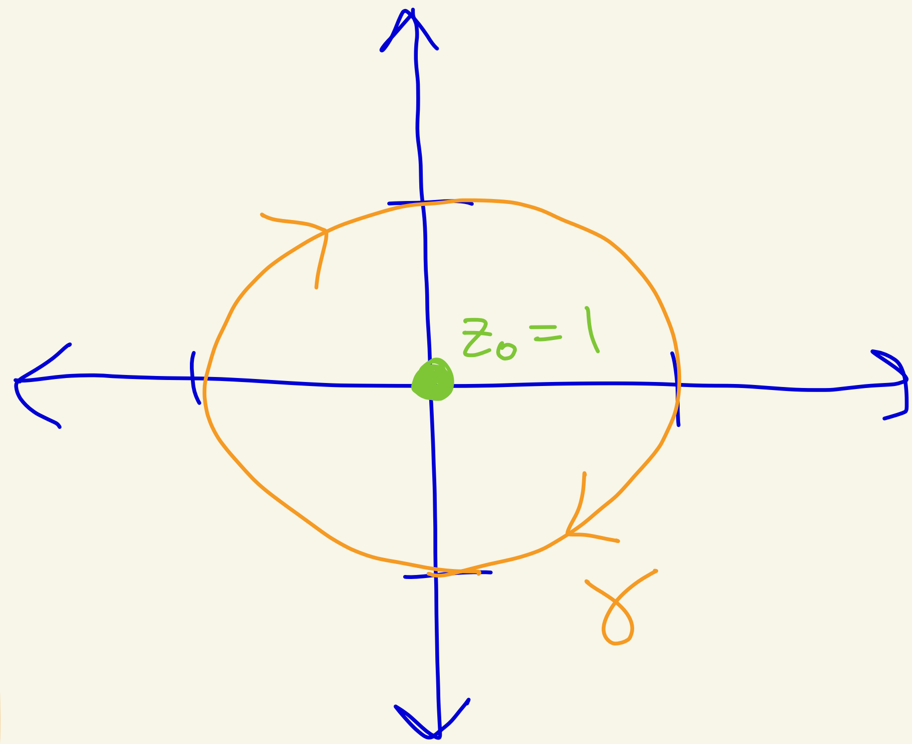
$$\int_\gamma \frac{z^2}{z-1}\,dz = \underbrace{2\pi i\,(1)^2}_{2\pi i f(z_0)} = 2\pi i$$

$z_0 = 1$

$f(z) = z^2$;

Cauchy integral formula

① (b)



γ is oriented clockwise

$-\gamma$ is oriented counter-clockwise

$$\int_{\gamma} \frac{\sin(z)}{z^2}\, dz = \underbrace{-\int_{-\gamma} \frac{\sin(z)}{(z-0)^2}\, dz}_{\text{change to counter-clockwise orientation}}$$

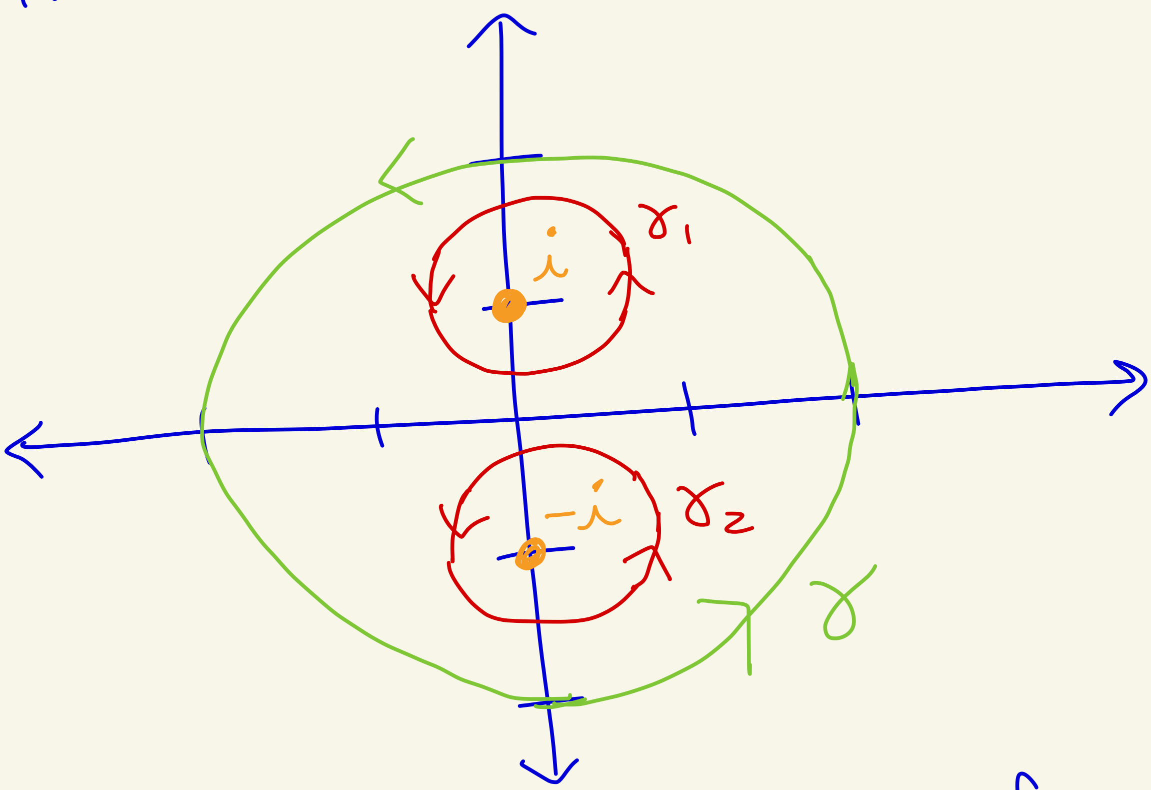
$$= -\frac{2\pi i}{1!} \underbrace{\cos(0)}_{f'(0)} = -2\pi i$$

$f(z) = \sin(z)$

$z_0 = 0,\ k = 1$

① (c)

We are integrating $\frac{z^2-1}{z^2+1}$. Note that $z^2+1=0$ when $z=\pm i$.



Let γ_1 be the circle of radius $\frac{1}{2}$ centered at i and γ_2 be the circle of $1/2$ centered at $-i$, both oriented counterclockwise. Then since $(z^2-1)/(z^2+1)$ is analytic on and between γ and γ_1, γ_2 we have

$$\int_{\gamma} \frac{z^2-1}{z^2+1}\,dz = \int_{\gamma_1} \frac{z^2-1}{z^2+1}\,dz + \int_{\gamma_2} \frac{z^2-1}{z^2+1}\,dz$$

Since $\dfrac{z^2-1}{z^2+1} = \dfrac{z^2-1}{(z-i)(z+i)}$ we have

$$\int_\gamma \frac{z^2-1}{z^2+1}\,dz = \underbrace{\int_{\gamma_1} \frac{\left(\frac{z^2-1}{z+i}\right)}{z-i}\,dz}_{z_0 = i} + \underbrace{\int_{\gamma_2} \frac{\left(\frac{z^2-1}{z-i}\right)}{z+i}\,dz}_{z_0 = -i}$$

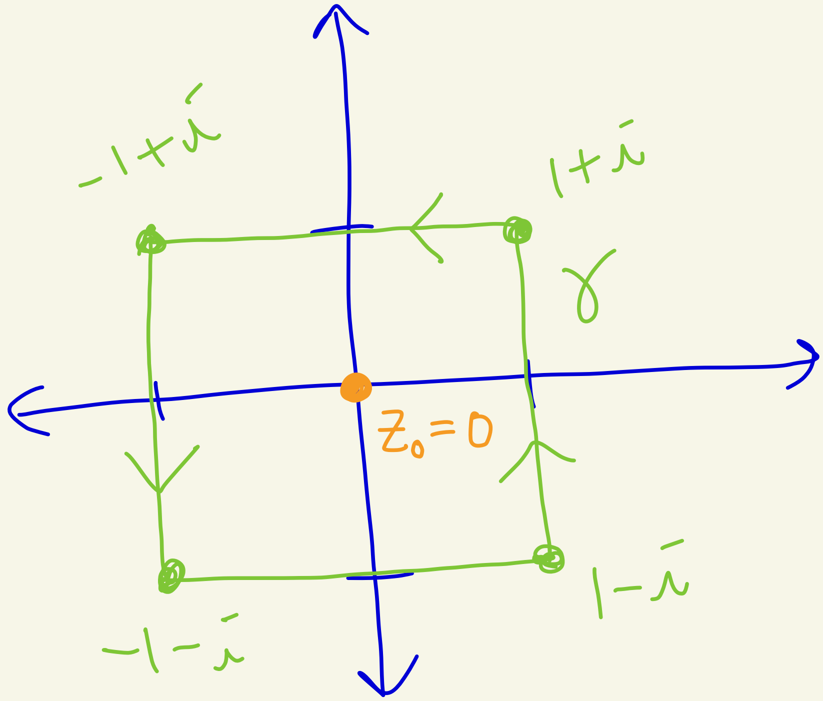
$$= 2\pi i\left(\frac{(i)^2-1}{i+i}\right) + 2\pi i\left(\frac{(-i)^2-1}{-i-i}\right)$$

Cauchy Integral Formula

$$= 2\pi i\left(\frac{-2}{2i}\right) + 2\pi i\left(\frac{-2}{-2i}\right)$$

$$= -2\pi + 2\pi = 0$$

1 (d)



$$\int_\gamma \frac{z^{10}+5z^3+1}{z^4}\,dz = \int \frac{z^{10}+5z^3+1}{(z-0)^4}\,dz$$

$$= \underbrace{\frac{2\pi i}{3!}\left[720(0)^7+30\right]}$$

$f(z) = z^{10}+5z^3+1$

$z_0 = 0$

$k = 3$

$\boxed{\frac{2\pi i}{3!} f^{(3)}(0)}$

Cauchy integral thm

$f'(z) = 10z^9 + 15z^2$

$f''(z) = 90z^8 + 30z$

$f'''(z) = 720z^7 + 30$

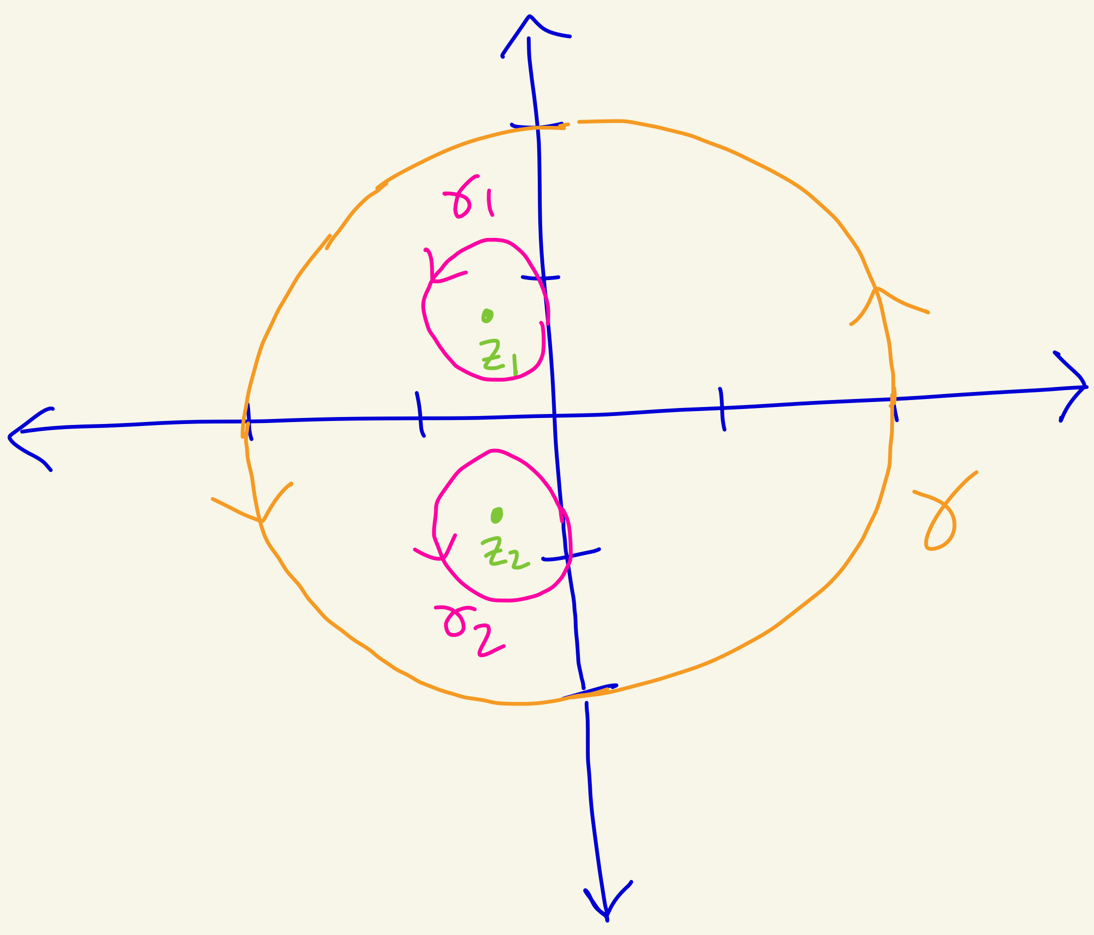
$$= \frac{60\pi i}{6} = 10\pi i$$

1 (e) $z^2 + z + 1 = 0$

iff $z = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \dfrac{-1 \pm \sqrt{-3}}{2}$

$= \underbrace{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}_{z_1}, \underbrace{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}_{z_2}$

Note that $\frac{\sqrt{3}}{2} \approx 0.866$



Since $\dfrac{1}{(z^2+z+1)^2}$ is analytic on and between γ and γ_1, γ_2 we have that

$$\int_\gamma \frac{dz}{(z^2+z+1)^2} = \int_{\gamma_1} \frac{dz}{(z^2+z+1)^2} + \int_{\gamma_2} \frac{dz}{(z^2+z+1)^2}$$

$$= \int_{\gamma_1} \frac{dz}{(z-z_1)^2(z-z_2)^2} + \int_{\gamma_2} \frac{dz}{(z-z_1)^2(z-z_2)^2}$$

$$= \int_{\gamma_1} \frac{\overbrace{\dfrac{1}{(z-z_2)^2}}^{f}}{(z-z_1)^2}\, dz + \int_{\gamma_2} \frac{\overbrace{\dfrac{1}{(z-z_1)^2}}^{f}}{(z-z_2)^2}\, dz$$

$$= \underbrace{\frac{2\pi i}{1!}\left[\overbrace{-2(z_1-z_2)^{-3}}^{f'(z_1)}\right]}_{\substack{f(z) = (z-z_2)^{-2} \\ f'(z) = -2(z-z_2)^{-3}}} + \underbrace{\frac{2\pi i}{1!}\left[\overbrace{-2(z_2-z_1)^{2}}^{f'(z_2)}\right]}_{\substack{f(z) = (z-z_1)^{-2} \\ f'(z) = -2(z-z_1)^{-3}}}$$

$$z_1 - z_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \sqrt{3} i$$

$$z_2 - z_1 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = -\sqrt{3} i$$

$\int_{0}$,

$$\int_\gamma \frac{dz}{(z^2+z+1)^2} = 2\pi i \left[-2(\sqrt{3} i)^{-3}\right] + 2\pi i \left[-2(-\sqrt{3} i)^{-3}\right]$$

$$= \frac{-4\pi i}{\sqrt{3}^3 i^3} + \frac{-4\pi i}{(-\sqrt{3})^3 (i)^3}$$

$$= \frac{-4\pi i}{3\sqrt{3}(-i)} + \frac{4\pi i}{3\sqrt{3}(-i)}$$

$$= \frac{4\pi}{3\sqrt{3}} + \frac{-4\pi}{3\sqrt{3}} = 0$$

$\boxed{1(f)}$ Note that

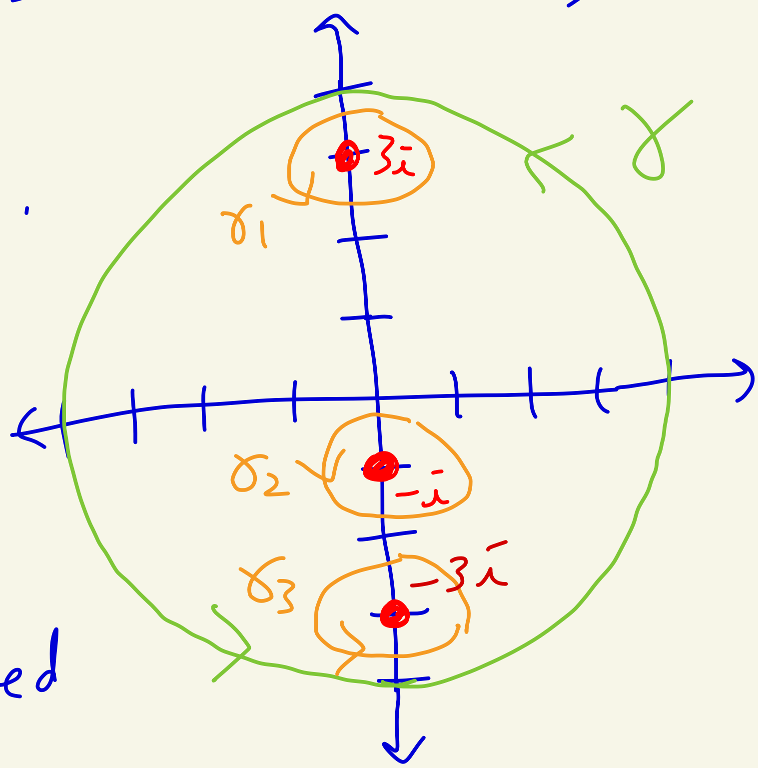
the roots of $z^2+9=0$ are $z=\pm 3i$

$$\frac{z}{(9+z^2)(z+i)^2} = \frac{z}{(z+3i)(z-3i)(z+i)^2}$$

So, inside γ we have three points where

$$f(z) = \frac{z}{(9+z^2)(z+i)^2}$$ is not analytic,

ie at $z = 3i, -3i, -i$.



Let $\gamma_1, \gamma_2, \gamma_3$ be circles of radius $1/2$ centered at $3i, -i, -3i$ respectively, each oriented counterclockwise.

Since $f(z)$ is analytic on and between γ and $\gamma_1, \gamma_2, \gamma_3$ we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$

So,

$$\int_{\gamma} \frac{z}{(9+z^2)(z+i)^2}\,dz = \int_{\gamma_1} \frac{\left[\frac{z}{(z+3i)(z+i)^2}\right]}{(z-3i)}\,dz \quad ①$$

$$+ \int_{\gamma_2} \frac{\left[z/(z+3i)(z-3i) \right]}{(z+i)^2}\,dz \quad ②$$

$$+ \int_{\gamma_3} \frac{\left[z/(z-3i)(z+i)^2 \right]}{(z+3i)}\,dz \quad ③$$

Calculating ①, by Cauchy's Integral Formula we get

$$\int_{\gamma_1} \frac{\left[\frac{z}{(z+3i)(z+i)^2}\right]}{(z-3i)} = 2\pi i \left[\frac{3i}{(3i+3i)(3i+i)^2}\right]$$

$$= 2\pi i \left[\frac{3i}{(6i)(4i)^2}\right] = \frac{6\pi i}{6(4i)^2} = \frac{-\pi i}{16}$$

Calculating 2 and letting $g(z) = \frac{z}{9+z^2}$

we have

$$\int_{\gamma_2} \frac{\left[z / (z+3i)(z-3i) \right]}{(z+i)^2} \, dz = \int_{\gamma_2} \frac{g(z)}{(z+i)^2}$$

$$= \frac{2\pi i}{1!} g'(-i) = 2\pi i \left[\frac{9-(-i)^2}{(9+(-i)^2)^2} \right] = 2\pi i \left[\frac{10}{(8)^2} \right]$$

Cauchy integral formula

$$g'(z) = \frac{(1)(9+z^2)-(z)(2z)}{(9+z^2)^2} = \frac{9-z^2}{(9+z^2)^2}$$

$$= 2\pi i \left[\frac{10}{64} \right] = 2\pi i \left[\frac{5}{32} \right] = \frac{5}{16}\pi i$$

Calculating 3 we have that

Cauchy integral formula

$$\int_{\gamma_3} \frac{\left[z / (z-3i)(z+i)^2 \right]}{(z+3i)} \, dz = 2\pi i \left[\frac{-3i}{(-3i-3i)(-3i+i)^2} \right]$$

$$= 2\pi i \left[\frac{-3i}{(-6i)(-2i)^2} \right] = \frac{\pi i}{(-2i)^2} = \frac{-\pi i}{4}$$

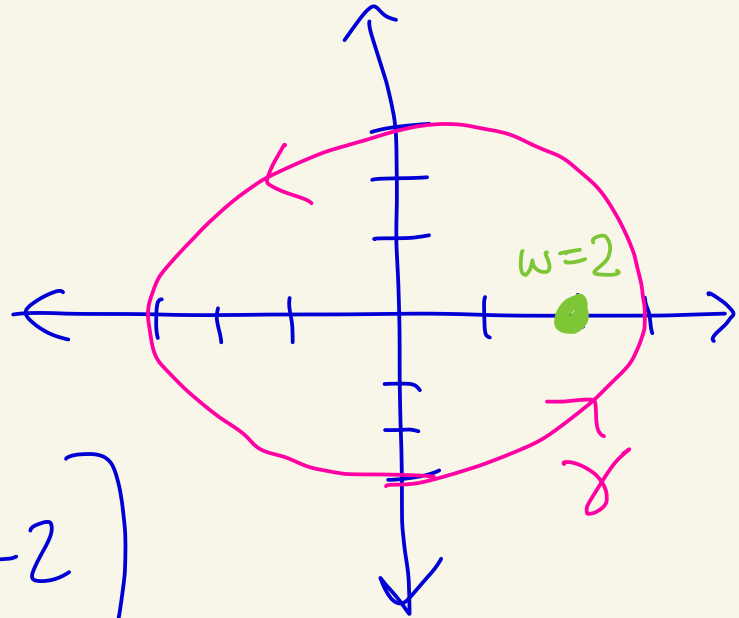
So,

$$\int_\gamma \frac{z}{(9+z^2)(z+i)^2}\, dz = \overbrace{\frac{-\pi i}{16} + \frac{5}{16}\pi i - \frac{\pi i}{4}}^{1 + 2 + 3}$$

$$= \frac{-\pi i + 5\pi i - 4\pi i}{16} = 0$$

2 (a)

$$g(2) = \int_\gamma \frac{2z^2 - z - 2}{z - 2}$$



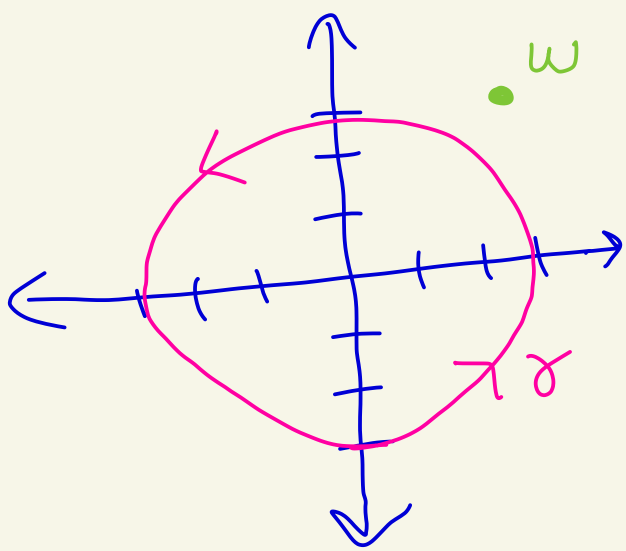
$$= 2\pi i \left[2(2)^2 - (2) - 2 \right]$$

Cauchy integral formula

$$= 2\pi i \left[4 \right] = 8\pi i$$

---

2 (b) Let $w \in \mathbb{C}$, with $|w| > 3$. Then w lies outside of γ. So,

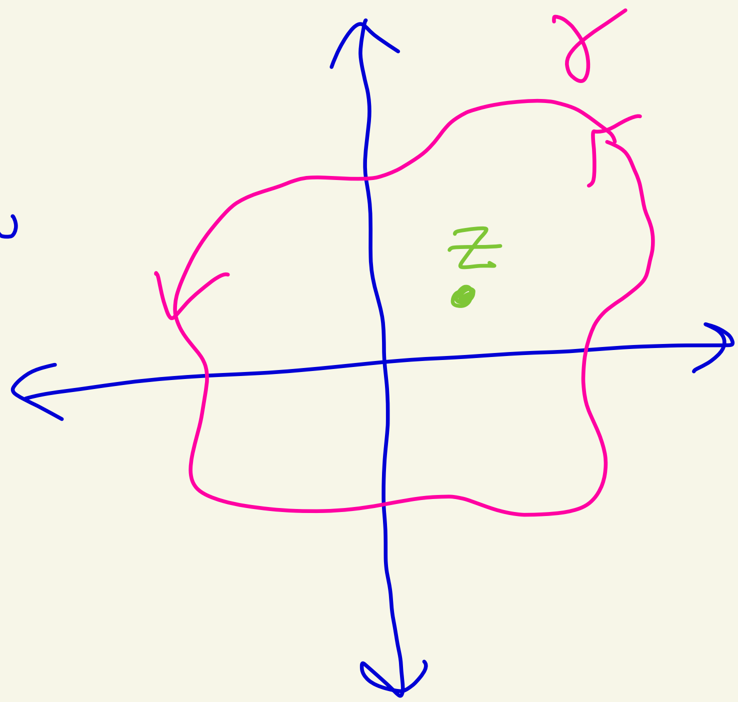


$$f(z) = \frac{2z^2 - z - 2}{z - 2}$$ is analytic in and on γ.

By Cauchy's theorem

$$\int_\gamma \frac{2z^2 - z - 2}{z - 2} \, dz = 0$$

3 Let z be inside of γ.
Then since f is analytic inside and on γ, by the cauchy integral formula we have



$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{0}{w-z}\,dw$$

$$= 0$$

$\boxed{\text{since } f(w)=0 \text{ for all } w \text{ on } \gamma}$

Thus, $f(z)=0$ for all z inside of γ.