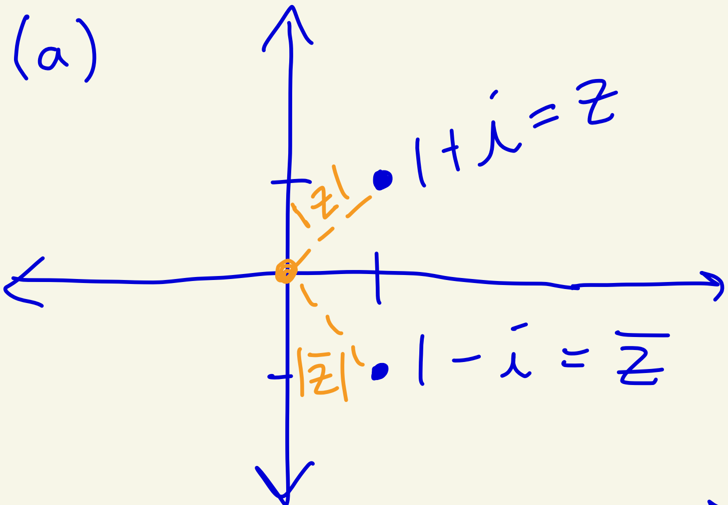
4680 - HW 1

Solutions

HW 1 Solutions

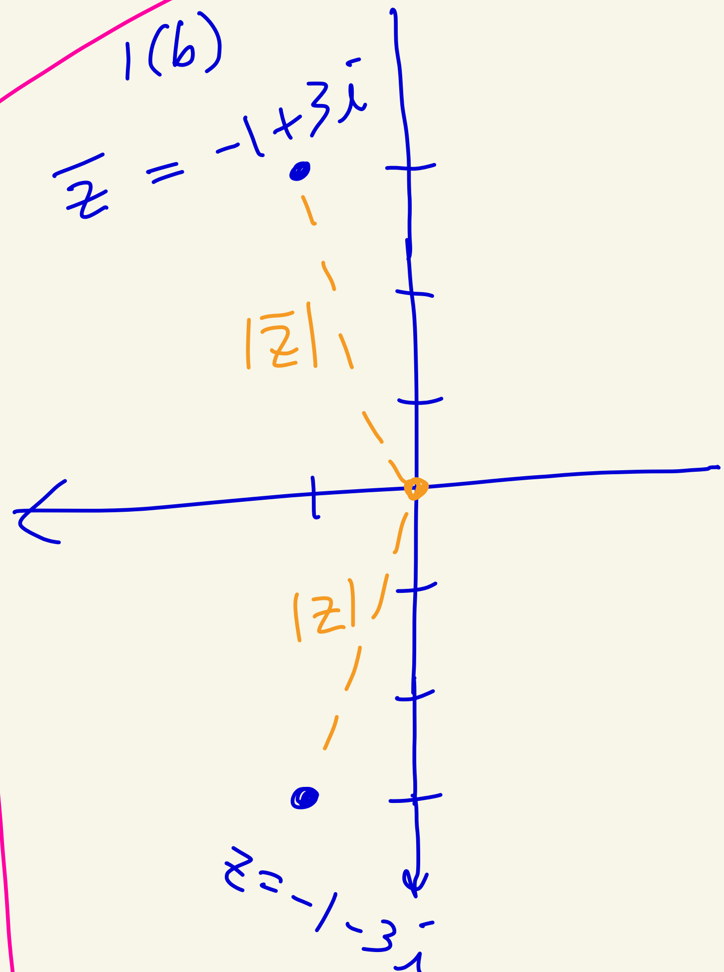
1(a)



$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$|\bar{z}| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

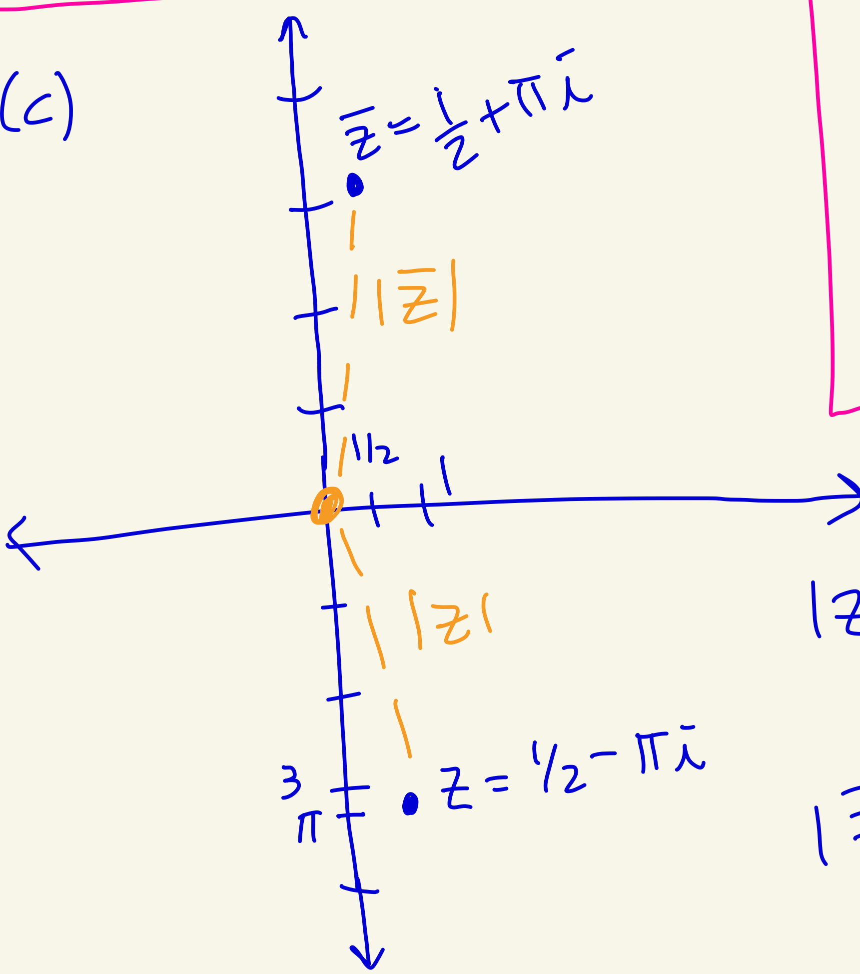
1(b)



$$|z| = \sqrt{(-1)^2+(-3)^2} = \sqrt{10}$$

$$|\bar{z}| = \sqrt{(-1)^2+3^2} = \sqrt{10}$$

1(c)



$$|z| = \sqrt{\left(\tfrac{1}{2}\right)^2 + (-\pi)^2} = \sqrt{\pi^2 + 1/4}$$

$$|\bar{z}| = \sqrt{(1/2)^2 + \pi^2} = \sqrt{\pi^2 + 1/4}$$

$\overline{w}$

2(a) $\dfrac{2+3i}{4+i} = \dfrac{2+3i}{4+i} \cdot \dfrac{4-i}{4-i} = \dfrac{8-2i+12i-3i^2}{16-4i+4i-i^2}$

w

$i^2=-1$

$$= \frac{8+10i+3}{16+0+1}$$

$$= \frac{11+10i}{17}$$

$$= \boxed{\frac{11}{17} + \frac{10}{17}i}$$

2(b) $(\sqrt{2}-i)(1-i\sqrt{2}) = \sqrt{2} - i\sqrt{2}\sqrt{2} - i + i^2\sqrt{2}$

$i^2=-1$

$$= \sqrt{2} - 2i - i + (-1)\sqrt{2}$$

$$= 0 - 3i$$

$$= \boxed{-3i}$$

2(c) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = \dfrac{1+2i}{3-4i} \cdot \dfrac{3+4i}{3+4i} + \dfrac{2-i}{5i} \cdot \dfrac{-5i}{-5i}$

$$= \frac{3+4i+6i-8}{9+12i-12i+16} + \frac{-10i+5i^2}{-25i^2} = \frac{-5+10i}{25} + \frac{-5-10i}{25}$$

$i^2=-1$

$$= \frac{-10}{25} = \boxed{-\frac{2}{5}}$$

2(d) $(1-i)^4 = \left((1-i)^2\right)^2 = \left(1-2i+i^2\right)^2$

$= (1-2i-1)^2 = (-2i)^2 = 4i^2$

$= -4$

2(e) $\left(2+\frac{1}{1-i}\right)^2 = \left(2+\frac{1}{1-i}\cdot\frac{1+i}{1+i}\right)^2$

$= \left(2+\frac{1+i}{1+i-i-i^2}\right)^2 = \left(2+\frac{1+i}{2}\right)^2$

$= \left(2+\frac{1}{2}+\frac{1}{2}i\right)^2 = \left(\frac{5}{2}+\frac{1}{2}i\right)^2$

$= \frac{25}{4}+\frac{5}{4}i+\frac{5}{4}i+\frac{1}{4}i^2$

$= \frac{25}{4}-\frac{1}{4}+\frac{10}{4}i = 6+\frac{5}{2}i$

3(a) $$\left|\frac{i(2+4i)(1-2i)}{(2-i)}\right| = \frac{|i||2+4i||1-2i|}{|2-i|}$$

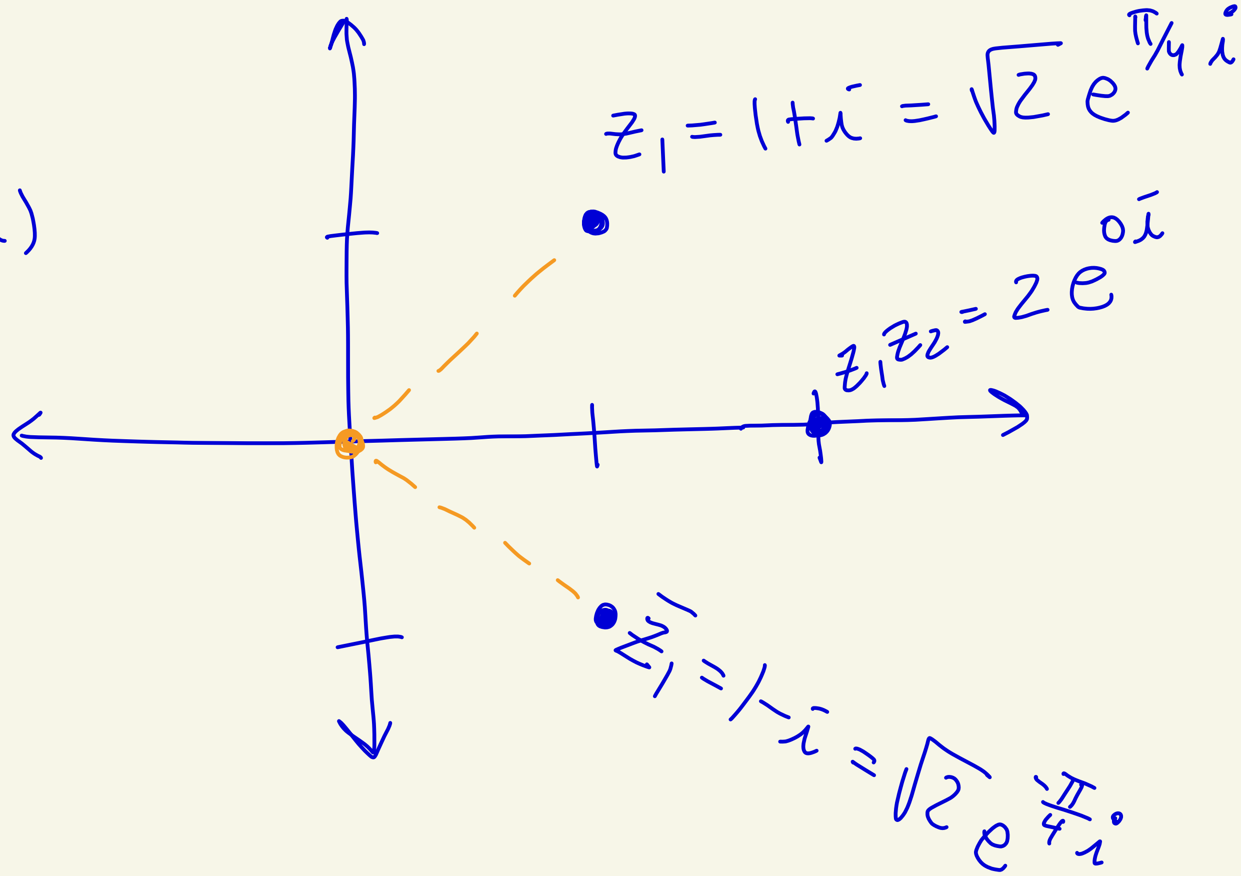
$$= \frac{\sqrt{1^2}\cdot\sqrt{2^2+4^2}\,\sqrt{1^2+(-2)^2}}{\sqrt{2^2+(-1)^2}} = \frac{\sqrt{20}\,\sqrt{5}}{\sqrt{5}}$$

$$= \sqrt{20}$$

3(b) $$\left|\frac{(3i)^2}{(-3+i)^6}\right| = \frac{|3i|^2}{|-3+i|^6} = \frac{\left(\sqrt{0^2+3^2}\right)^2}{\left(\sqrt{(-3)^2+1^2}\right)^6}$$

$$= \frac{(\sqrt{9})^2}{(\sqrt{10})^6} = \frac{9}{10^3} = \frac{9}{1000}$$

4(a)



$|z_1| = \sqrt{2}$

$z_1 = re^{i\theta} = \sqrt{2}\, e^{\pi/4\, i}$

$|z_2| = \sqrt{2}$

$z_2 = re^{i\theta} = \sqrt{2}\, e^{-\pi/4\, i}$

add the angles

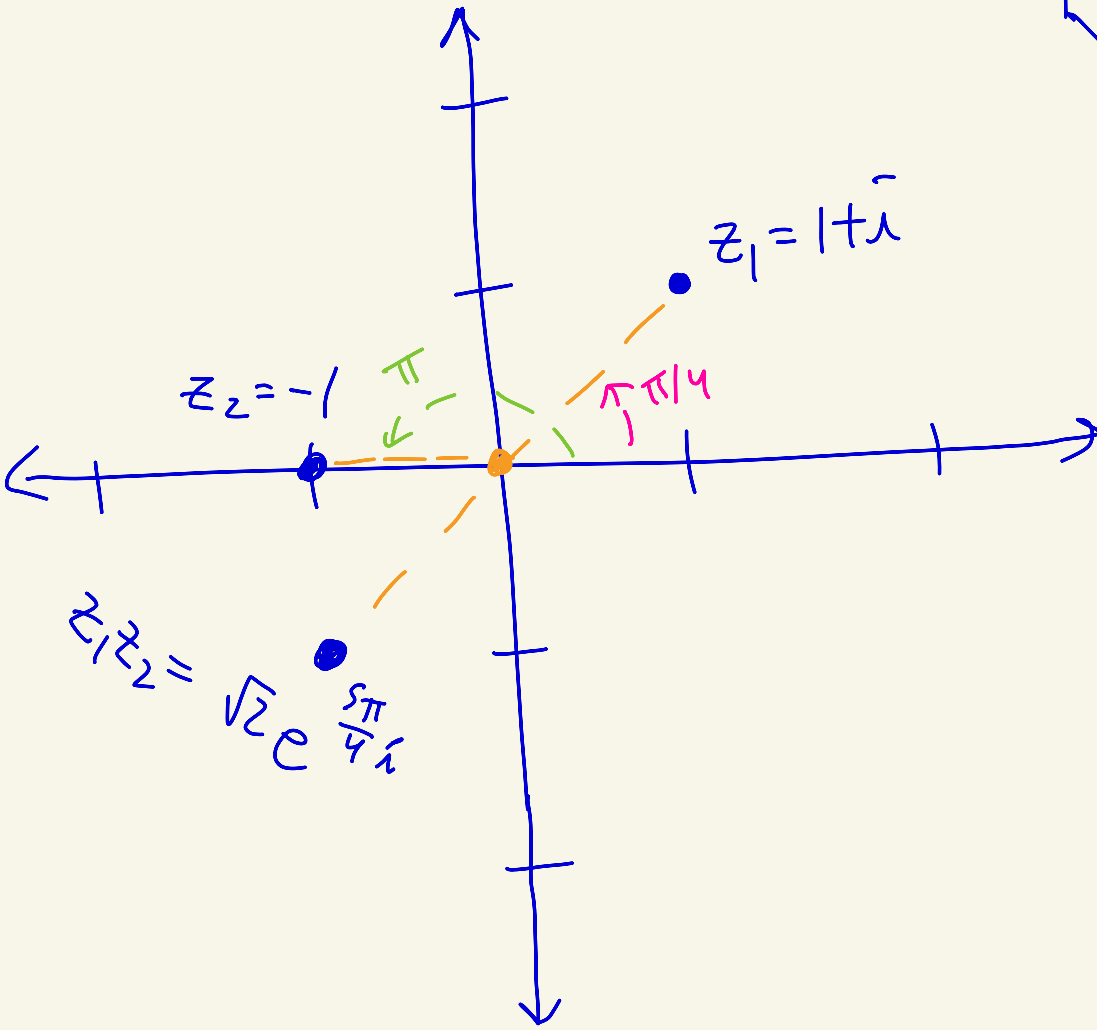
$z_1 z_2 = \sqrt{2}\, e^{\pi/4\, i} \sqrt{2}\, e^{-\pi/4\, i} = 2\, e^{\left(\frac{\pi}{4} - \frac{\pi}{4}\right) i}$

$= 2e^{0i}$

multiply the lengths

So, $z_1 z_2$ has length $r = 2$ and angle $\theta = 0$

4(b)



$\underline{z_1}$

$r = |z_1| = |1+i| = \sqrt{2}$

$\theta = \pi/4$

$z_1 = \sqrt{2}\, e^{(\pi/4) i}$

$\underline{z_2}$

$r = |z_2| = |-1| = 1$

$\theta = \pi$

$z_2 = 1 \cdot e^{\pi i} = e^{\pi i}$

$\underline{z_1 z_2}$

$$z_1 z_2 = \left(\sqrt{2}\, e^{(\pi/4) i}\right)\left(e^{\pi i}\right) = \sqrt{2}\, e^{\left(\frac{\pi}{4} + \pi\right) i}$$

$$= \sqrt{2}\, e^{(5\pi/4) i}$$

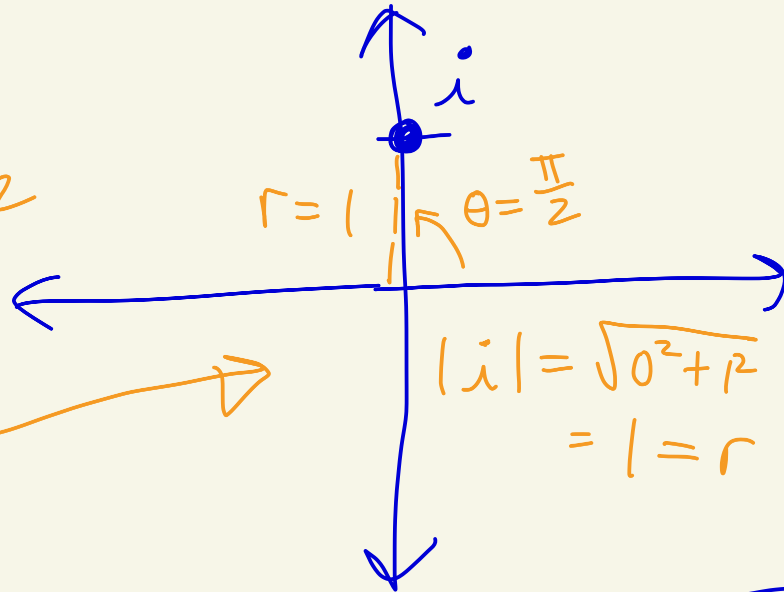
In non-polar form $z_1 z_2 = \sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right] = \sqrt{2}\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right] = -1 - i$

5(a)

$$z^2 - i = 0$$

$$z^{2} = i \quad \xleftarrow{\ n=2\ }$$

$$i = 1 \cdot e^{\frac{\pi}{2} i}$$



$$z^2 = 1 \cdot e^{\frac{\pi}{2} i}$$

$$\boxed{z_k = r^{1/n} e^{\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) i}} \quad k = 0, 1, \ldots, n-1$$

$$z_k = 1^{1/2} \cdot e^{\left(\frac{(\pi/2)}{2} + \frac{2\pi k}{2}\right) i} \qquad k = 0, 1$$

$$z_0 = e^{(\pi/4) i} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

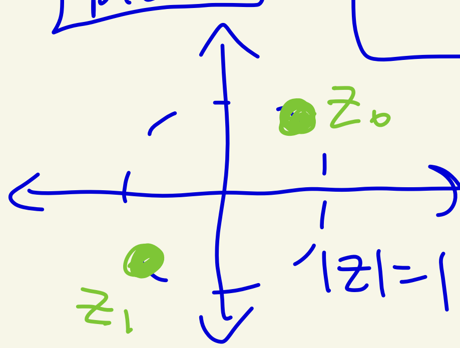
$$z_1 = e^{(\pi/4 + \pi) i} = e^{(5\pi/4) i} = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)$$
$$= -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}$$

Answer:

$$z_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

$$z_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

picture



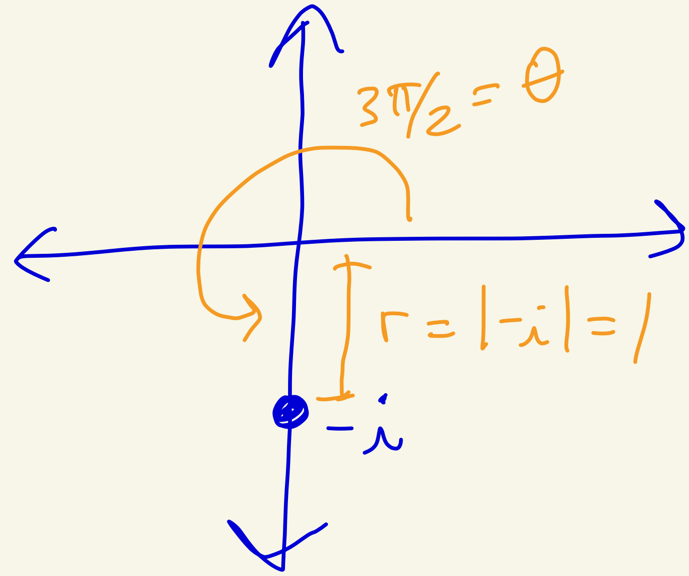
(on unit circle)

5(b)

$$z^4 + i = 0$$

$$z^4 = -i$$

$$-i = 1 \cdot e^{\frac{3\pi}{2} i}$$



$$z^4 = 1 \cdot e^{\frac{3\pi}{2} i}$$

$$z_k = 1^{1/4} e^{\left(\frac{3\pi/2}{4} + \frac{2\pi k}{4}\right) i} \qquad k = 0, 1, 2, 3$$

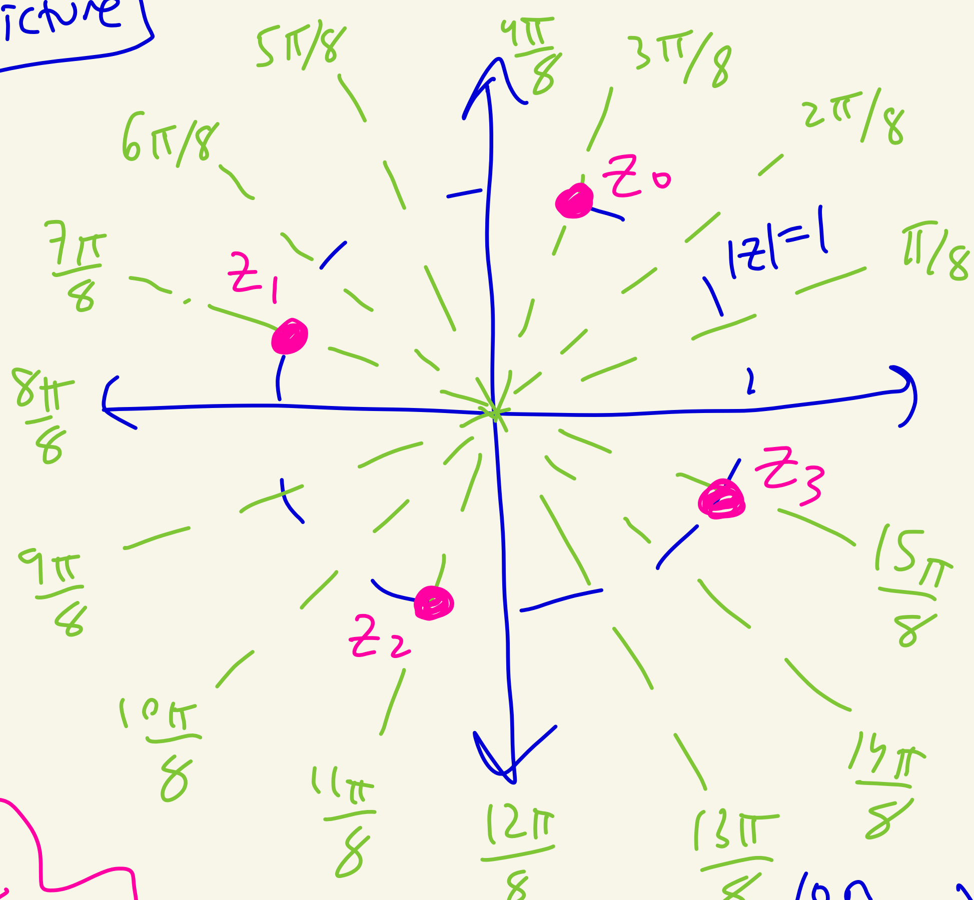
picture

$$z_0 = e^{(3\pi/8) i}$$

$$z_1 = e^{(7\pi/8) i}$$

$$z_2 = e^{(11\pi/8) i}$$

$$z_3 = e^{(15\pi/8) i}$$



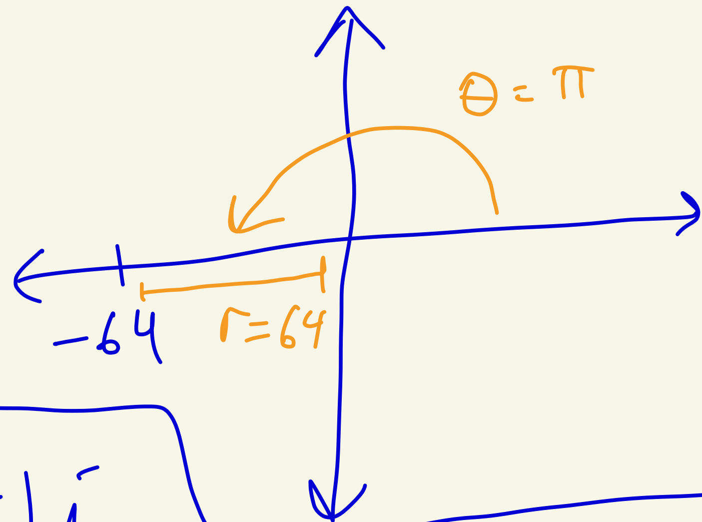
(on unit circle)

can just leave answer like this since can't calculate sin/cos of these angles exactly

5(c)

$$z^6 = -64$$

$$-64 = 64 \cdot e^{\pi i}$$



$$z_k = 64^{1/6} e^{\left(\frac{\pi}{6} + \frac{2\pi k}{6}\right) i}, \quad k = 0, 1, 2, 3, 4, 5$$

$$z_0 = 2e^{(\pi/6) i} = 2\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]$$
$$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right) = \sqrt{3} + i$$

$$z_1 = 2e^{\left(\frac{\pi}{2}\right) i} = 2\left[0 + 1 \cdot i\right] = 2i$$

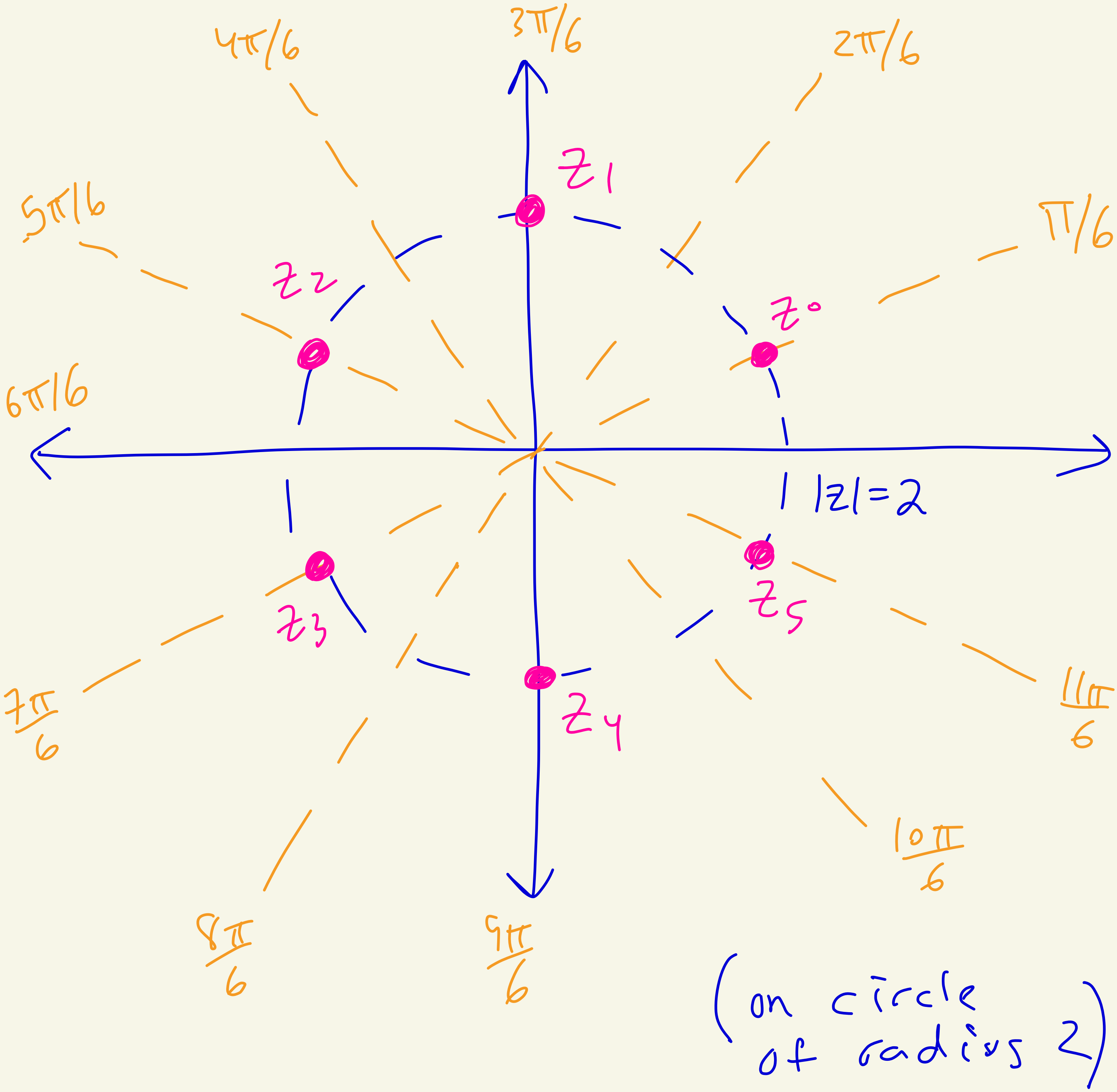
$$z_2 = 2e^{\frac{5\pi}{6} i} = 2\left[-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right] = -\sqrt{3} + i$$

$$z_3 = 2e^{\frac{7\pi}{6} i} = 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right] = -\sqrt{3} - i$$

$$z_4 = 2e^{\frac{3\pi}{2} i} = 2\left[0 - 1 \cdot i\right] = -2i$$

$$z_5 = 2e^{\frac{11\pi}{6} i} = 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2} i\right] = \sqrt{3} - i$$

[picture on next page]



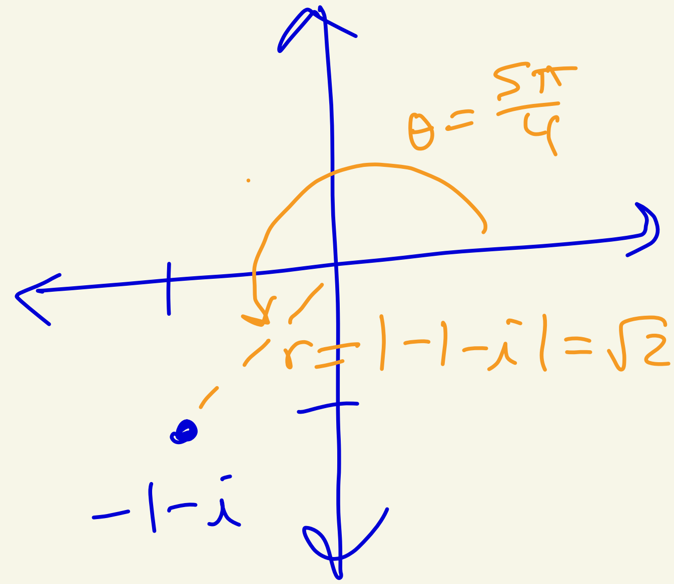
3π/6
4π/6
2π/6
5π/6
π/6
6π/6
z_0
z_1
z_2
z_3
z_4
z_5
|z|=2
7π/6
11π/6
10π/6
8π/6
9π/6
(on circle of radius 2)

5(d)

$$z^3 + (1+i) = 0$$

$$z^3 = -1-i$$

$$-1-i = \sqrt{2}\, e^{\frac{5\pi}{4} i}$$



$$z_k = \left(\sqrt{2}\right)^{1/3} e^{\left(\frac{5\pi/4}{3} + \frac{2\pi k}{3}\right) i}$$

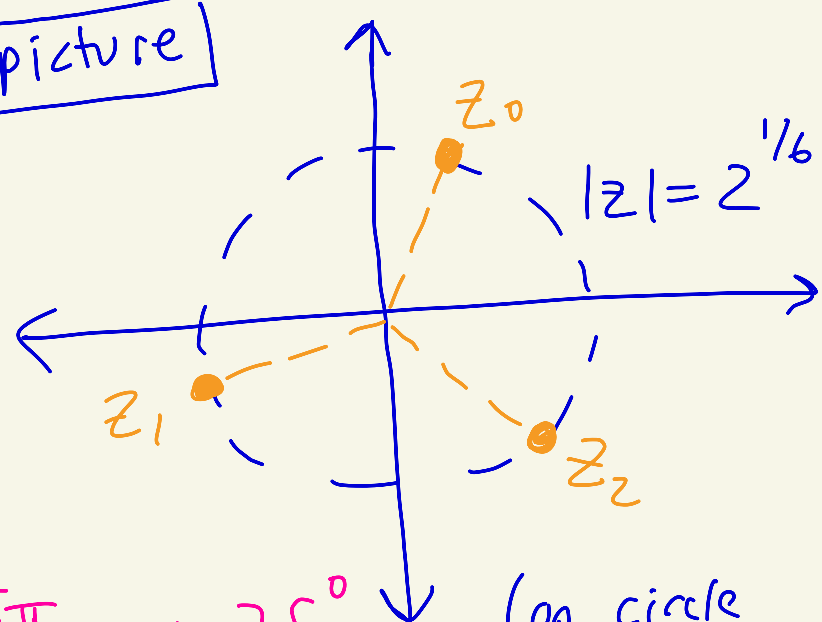
$$= 2^{1/6} e^{\left(\frac{5\pi}{12} + \frac{2\pi k}{3}\right) i} \qquad k = 0, 1, 2$$

$$z_0 = 2^{1/6} e^{\frac{5\pi}{12} i}$$

$$z_1 = 2^{1/6} e^{\frac{13\pi}{12} i}$$

$$z_2 = 2^{1/6} e^{\frac{21\pi}{12} i}$$

picture



$\frac{5\pi}{12} \longleftrightarrow 75^\circ$

$\frac{13\pi}{12} \longleftrightarrow 195^\circ$

$\frac{21\pi}{12} \longleftrightarrow 315^\circ$

(on circle of radius $2^{1/6}$)

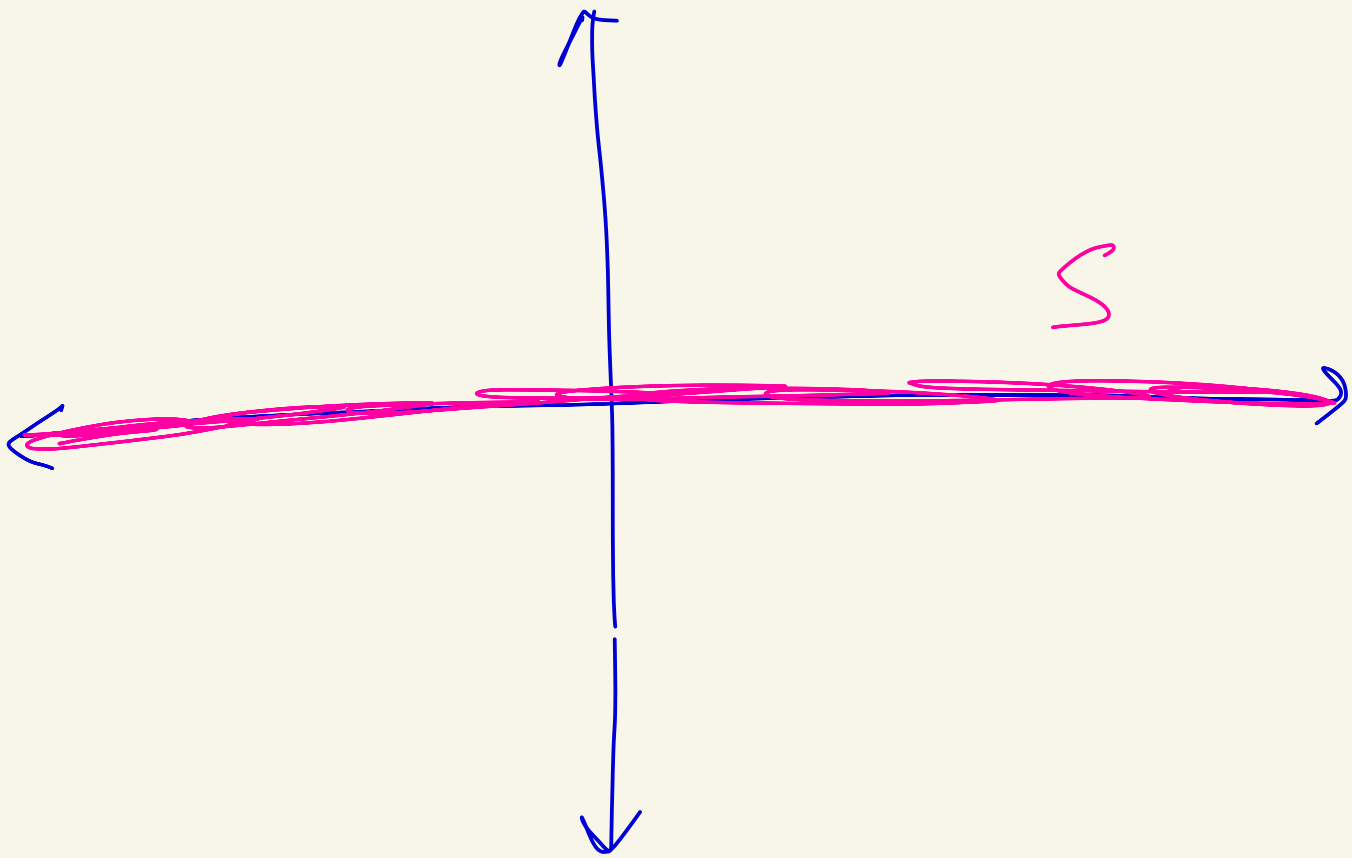
6(a)

$$S = \{ z \in \mathbb{C} \mid \operatorname{Im}(z+5) = 0 \}$$

If $z = x+iy$, then

$$\operatorname{Im}(z+5) = \operatorname{Im}(x+5+iy) = y$$

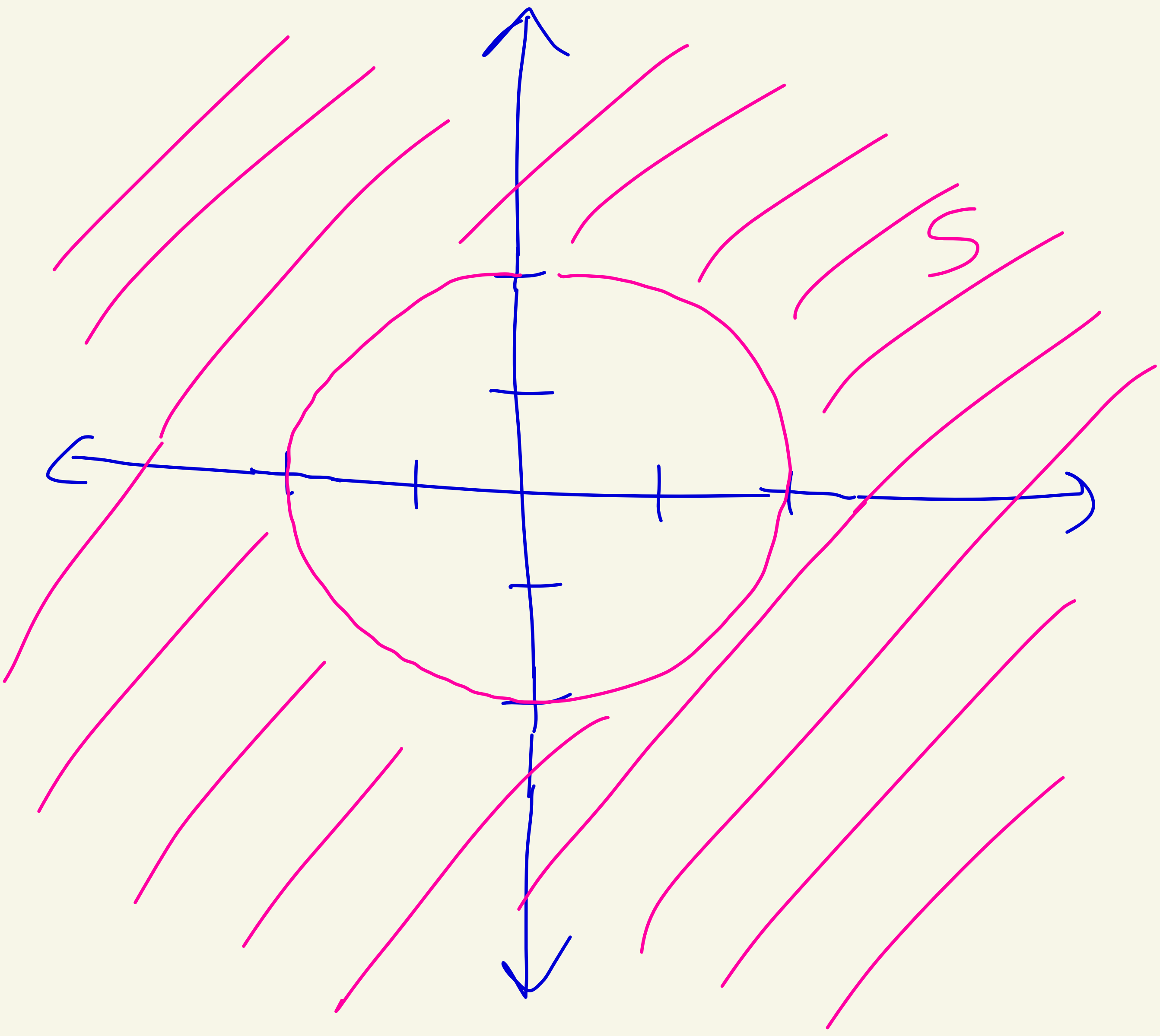
So, $S = \{ z = x+iy \in \mathbb{C} \mid y = 0 \}$



6 (b) $S = \{ z \in \mathbb{C} \mid |z^2| \geq 4 \}$

$= \{ z \in \mathbb{C} \mid |z|^2 \geq 4 \}$

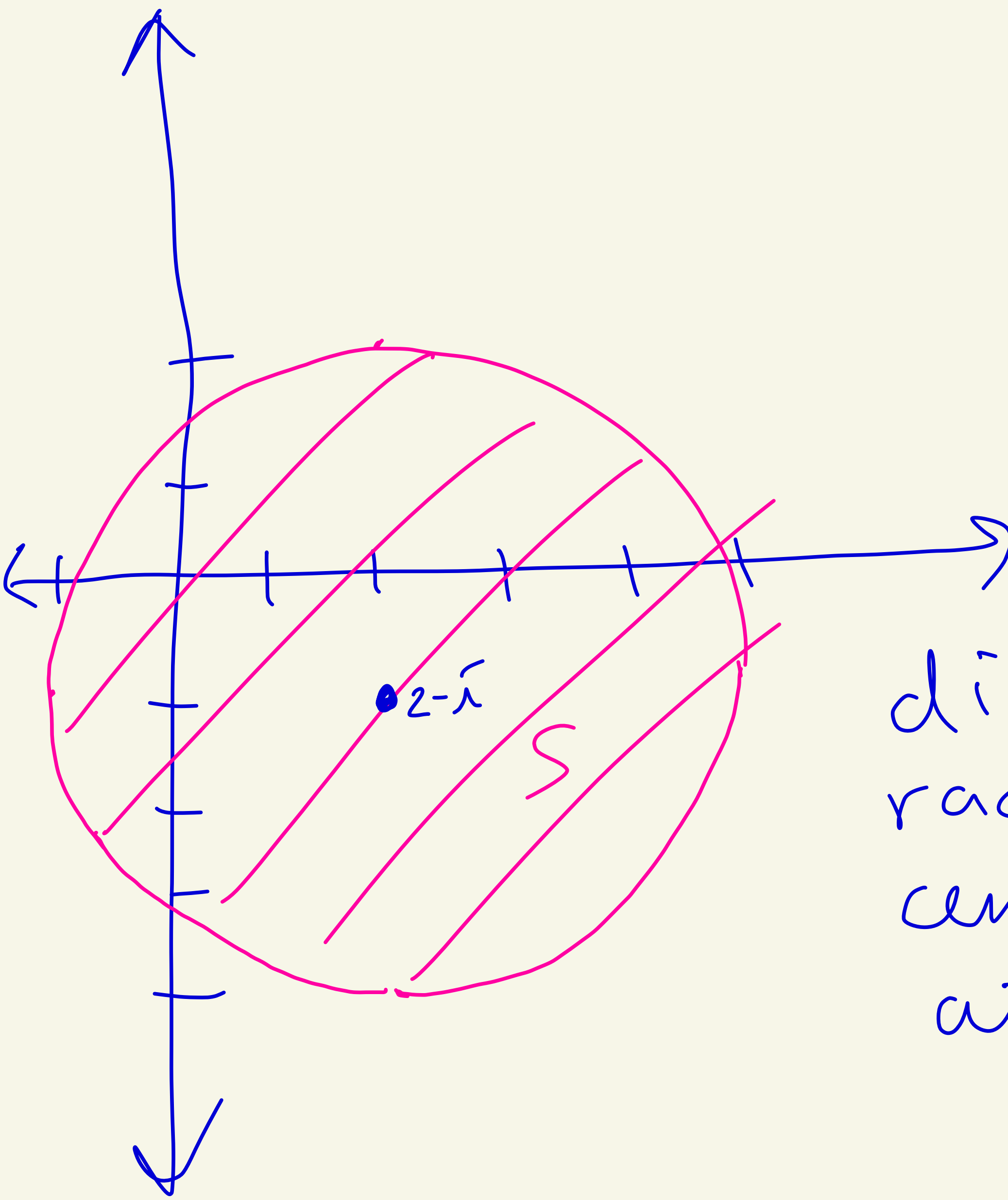
$= \{ z \in \mathbb{C} \mid |z| \geq 2 \}$



6(c)

$$S = \{ z \in \mathbb{C} \mid |z - 2 + i| \leq 3 \}$$
$$= \{ z \in \mathbb{C} \mid |z - (2 - i)| \leq 3 \}$$

$|z - c| \leq r$
disc of radius $r > 0$ centered at c.



disc of radius 3 centered at $2 - i$

6(d)

$$S = \left\{ z \in \mathbb{C} - \{0\} \;\middle|\; \mathrm{Re}\left(\frac{1}{z}\right) \geq \frac{1}{2} \right\}$$

Let $z = x + iy$. Then

$$\frac{1}{z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$$= \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right)$$

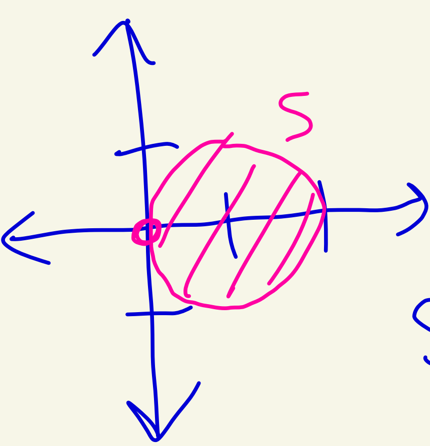
So, $S = \left\{ \substack{z = x+iy \\ z \neq 0} \;\middle|\; \frac{x}{x^2+y^2} \geq \frac{1}{2} \right\}$

$\frac{x}{x^2+y^2} \geq \frac{1}{2}$ iff $2x \geq x^2 + y^2$

iff $0 \geq x^2 + y^2 - 2x$

iff $0 \geq x^2 - 2x + 1 + y^2 - 1$

iff $1 \geq (x-1)^2 + y^2$



$$S = \left\{ \substack{z = x+iy \\ z \neq 0} \;\middle|\; (x-1)^2 + y^2 \leq 1 \right\}$$

7(a) $z = x + iy$

$$\frac{1}{z^2} = \frac{1}{(x^2-y^2)+i2xy}$$

$$= \left(\frac{1}{(x^2-y^2)+i2xy}\right)\cdot\left(\frac{(x^2-y^2)-i2xy}{(x^2-y^2)-i2xy}\right)$$

$$= \frac{(x^2-y^2)-i2xy}{(x^2-y^2)^2+(2xy)^2}$$

$$= \underbrace{\frac{x^2-y^2}{x^4+y^4+2x^2y^2}}_{Re\left(\frac{1}{z^2}\right)} + i\underbrace{\left(\frac{-2xy}{x^4+y^4+2x^2y^2}\right)}_{Im\left(\frac{1}{z^2}\right)}$$

$x^4 + y^4 - 2x^2y^2 + 4x^2y^2$

7(b) $z = x + iy$

$$\frac{z-1}{3z+2} = \frac{(x-1)+iy}{(3x+2)+i(3y)}$$

$$= \frac{(x-1)+iy}{(3x+2)+i(3y)} \cdot \frac{(3x+2)-i(3y)}{(3x+2)-i(3y)}$$

$$= \frac{3y^2+3x^2-2-x \; + i(5y)}{4+12x+9x^2+9y^2}$$

$$= \underbrace{\left(\frac{3y^2+3x^2-2-x}{4+12x+9x^2+9y^2}\right)}_{\mathrm{Re}\left(\frac{z-1}{3z+2}\right)} + i\underbrace{\left(\frac{5y}{4+12x+9x^2+9y^2}\right)}_{\mathrm{Im}\left(\frac{z-1}{3z+2}\right)}$$

(8) For all of 8, let $z = x + iy$ and $w = a + ib$.

8(a) $\overline{z+w} = \overline{(x+iy)+(a+ib)}$

$= \overline{(x+a)+i(y+b)}$

$= (x+a) - i(y+b)$

$= x - iy + a - ib$

$= \overline{z} + \overline{w}$

8(b) $\overline{zw} = \overline{(x+iy)(a+ib)}$

$= \overline{(xa-yb)+i(ya+xb)}$

$= (xa-yb) - i(ya+xb)$

$= (x-iy)(a-ib)$

8(c) $|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

$= (x+iy)(x-iy)$

$= z\,\overline{z}$

Notation from start of problem 8 is $z = x+iy$, $w = a+ib$

8(d) $|zw| = |(x+iy)(a+ib)|$

$= |(xa-yb)+i(ya+bx)|$

$= \sqrt{(xa-yb)^2 + (ya+bx)^2}$

$= \sqrt{x^2a^2 - 2xayb + y^2b^2 + y^2a^2 + 2xayb + b^2x^2}$

$= \sqrt{x^2a^2 + x^2b^2 + y^2b^2 + y^2a^2}$

$= \sqrt{(x^2+y^2)(a^2+b^2)}$

$= \sqrt{x^2+y^2}\sqrt{a^2+b^2} = |z|\,|w|$

Notation from start of 8 is $z = x+iy$
$w = a+ib$

8(e) Note that

$$\left|\frac{1}{w}\right| = \left|\frac{1}{a+ib}\cdot\frac{a-ib}{a-ib}\right| = \left|\frac{a-ib}{a^2+b^2}\right|$$

$$= \left|\frac{a}{a^2+b^2} + i\,\frac{-b}{a^2+b^2}\right| = \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2}$$

$$= \sqrt{\frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2}} = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}}$$

$$= \sqrt{\frac{1}{a^2+b^2}} = \frac{1}{\sqrt{a^2+b^2}} = \frac{1}{|w|}$$

So, $\boxed{\left|\frac{1}{w}\right| = \frac{1}{|w|}}$.

8(d)

Thus,

$$\left|\frac{z}{w}\right| = \left|z\cdot\frac{1}{w}\right| = |z|\left|\frac{1}{w}\right| = |z|\frac{1}{|w|}$$

$$= \frac{|z|}{|w|}$$

Notation from start of 8: $z = x + iy$

8(f)

$$\mathrm{Re}(iz) = \mathrm{Re}(i(x+iy))$$
$$= \mathrm{Re}(ix - y) = -y$$
$$= -\mathrm{Im}(x+iy) = -\mathrm{Im}(z)$$

$$\mathrm{Im}(iz) = \mathrm{Im}(i(x+iy))$$
$$= \mathrm{Im}(ix - y)$$
$$= x = \mathrm{Re}(z)$$

9. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ with $|z_3| \neq |z_4|$. So in particular $z_3 \neq -z_4$ and so $z_3 + z_4 \neq 0$.

From class,

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (*)$$

and

$$|z_3 + z_4| \geq \big|\,|z_3| - |z_4|\,\big|$$

So,

$$\frac{1}{|z_3+z_4|} \leq \frac{1}{\big|\,|z_3|-|z_4|\,\big|} \quad (**)$$

Thus,

$$\frac{|z_1+z_2|}{|z_3+z_4|} \overset{(*)}{\leq} \frac{|z_1|+|z_2|}{|z_3+z_4|} \overset{(**)}{\leq} \frac{|z_1|+|z_2|}{\big|\,|z_3|-|z_4|\,\big|}$$

10 (a) Let $z = x + iy$,

($\Rightarrow$) Suppose z is real.

Then $y = 0$,

So, $z = x + i0 = x - i0 = \overline{z}$

($\Leftarrow$) Suppose $z = \overline{z}$.

Then $x + iy = x - iy$.

So, $2iy = 0$.

Thus, $y = 0$,

So, $z = x + iy = x + 0i = x$

Thus, z is real.

10 (b) Let $z = x + iy$.

($\Rightarrow$) Suppose z is either real or pure imaginary.

Case 1: Suppose z is real.

Then $y = 0$.

So, $z = x + i0 = x$ and $\overline{z} = x - i0 = x$.

Thus, $(\overline{z})^2 = x^2 = z^2$

Case 2: Suppose z is pure imaginary

Then $x = 0$.

So, $z = iy$ and $\overline{z} = -iy$.

Thus, $(\overline{z})^2 = (-iy)^2 = (iy)^2 = z^2$.

In either case, $(\overline{z})^2 = z^2$

($\Leftarrow$) Now suppose that $(\bar{z})^2 = z^2$.

Then $(x - iy)^2 = (x + iy)^2$

So, $x^2 - 2ixy - y^2 = x^2 + 2ixy - y^2$.

Thus, $4ixy = 0$.

So, either $x = 0$ or $y = 0$.

If $x = 0$, then $z = x + iy = iy$
and z is pure imaginary.

If $y = 0$, then $z = x + iy = x$
and z is real.

(11) Suppose that $w \in \mathbb{C}$, $w^n = 1$, and $w \neq 1$.

Then,

$$1 + w + w^2 + \cdots + w^{n-1} = \frac{w^n - 1}{w - 1}$$

$$= \frac{1 - 1}{w - 1} = 0.$$

Geometric sum

$$1 + a + a^2 + \cdots + a^m = \frac{a^{m+1} - 1}{a - 1}$$

if $a \neq 1$

FOR FUN PROBLEM

A) (De Moivre's Formula)

We prove this by induction.

When $n=1$,

$$z = r\left[\cos(\theta) + i\sin(\theta)\right]$$
$$= r^1\left[\cos(1\cdot\theta) + i\sin(1\cdot\theta)\right].$$

Suppose $k \geq 1$ and

$$z^k = r^k\left[\cos(k\theta) + i\sin(k\theta)\right].$$

Then

$$z^{k+1} = z^k z$$
$$= \left[r^k\left[\cos(k\theta) + i\sin(k\theta)\right]\right]\left[r\left(\cos(\theta) + i\sin(\theta)\right)\right]$$
$$= r^{k+1}\left[\underbrace{\{\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)\}}_{\cos(k\theta+\theta)} + i\underbrace{\{\cos(k\theta)\sin(\theta) + \cos(\theta)\sin(k\theta)\}}_{\sin(k\theta+\theta)}\right]$$

$$= r^{k+1}\left[\cos(k\theta+\theta)+i\sin(k\theta+\theta)\right]$$

$$= r^{k+1}\left[\cos\left((k+1)\theta\right)+i\sin\left((k+1)\theta\right)\right]$$

Thus, by induction

$$z^n = r^n\left[\cos(n\theta)+i\sin(n\theta)\right]$$

for all $n \geq 1$.

For Fun Problem

B) Let $w = r[\cos(\theta) + i\sin(\theta)]$ where $r > 0$. We want to find all $z \in \mathbb{C}$ such that $z^n = w$.

Write $z = \underbrace{\rho[\cos(\psi) + i\sin(\psi)]}_{\text{where } \rho > 0}$,

We want to solve

$$\rho^n[\cos(n\psi) + i\sin(n\psi)] = r[\cos(\theta) + i\sin(\theta)]$$

So, $|\rho^n| \underbrace{|\cos(n\psi) + i\sin(n\psi)|}_{1} = |r| \underbrace{|\cos(\theta) + i\sin(\theta)|}_{1}$

Then $|\rho^n| = |r|$, Since $\rho > 0$, and $r > 0$ we get $\rho^n = r$. So, $\rho = r^{1/n}$.

Since $\rho^n = r$ we also get

$$\cos(\psi n) + i\sin(\psi n) = \cos(\theta) + i\sin(\theta).$$

So, $\cos(\psi n) = \cos(\theta)$

and $\sin(\psi n) = \sin(\theta)$.

These functions are 2π-periodic.

So, $\psi n = \theta + 2\pi k$ for some $k \in \mathbb{Z}$.

Thus, $\psi = \frac{\theta}{n} + \frac{2\pi k}{n}$ for some $k \in \mathbb{Z}$.

So,

$$z = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

Each value of k for $k = 0, 1, \ldots, n-1$ gives a different value for z but for other values of k we get repeats since sin/cos are 2π-periodic.