Homework #2 Solutions

4650 REVIEW

① Since $(b_n)_{n=1}^{\infty}$ is non-decreasing we have $b_n \leq b_{n+1}$ for all $n \geq 1$.

We will show that $b_n \leq L$ for all $n \geq 1$.

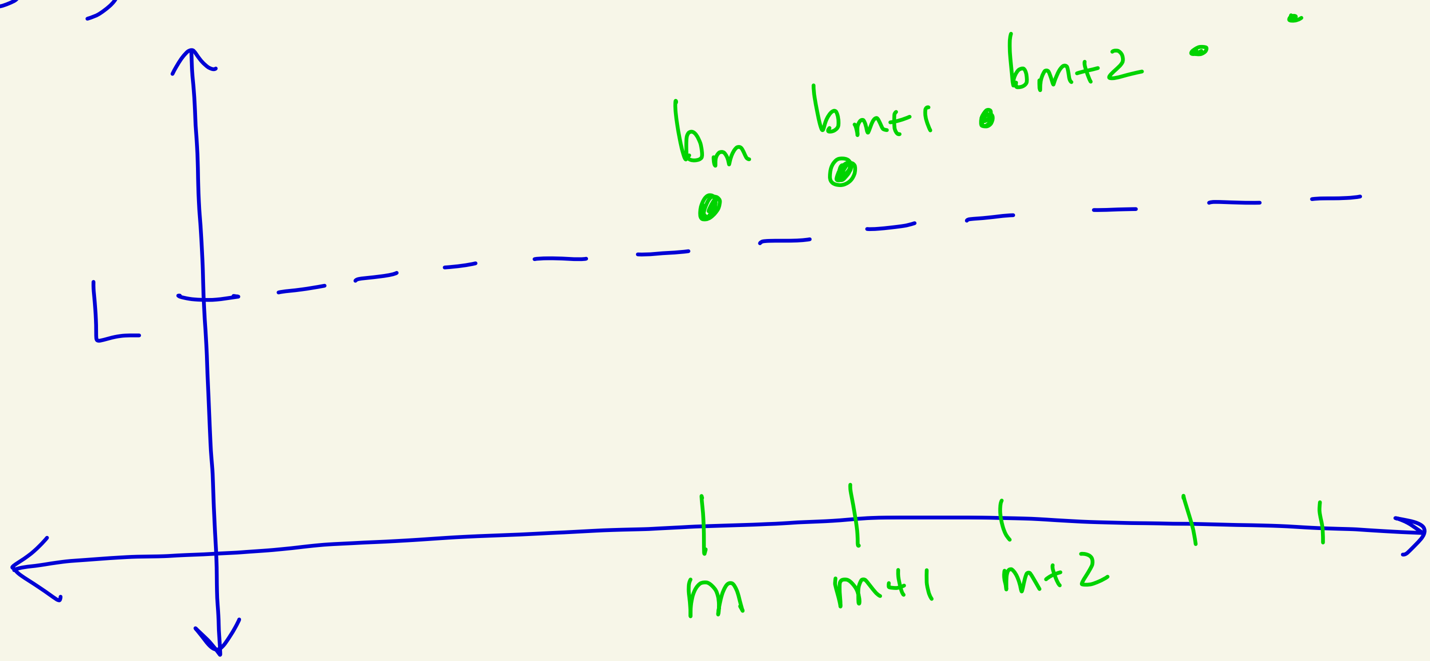
Suppose instead that $b_m > L$ for some $m \geq 1$.

Then,

$$L < b_m \leq b_{m+1} \leq b_{m+2} \leq \dots$$

That is, $L < b_m \leq b_n$ for all $n \geq m$.

So, $0 < b_m - L \leq b_n - L$ for all $n \geq m$.



Let $\varepsilon = b_m - L$.

Since $\lim_{n\to\infty} b_n = L$ we know

there exists $N > 0$ where

$|b_n - L| < \varepsilon$.

Let $\hat{N} = \max\{N, m\}$.

Let $n \geq \hat{N}$

Then, since $n \geq m$ we have

$$0 < \underbrace{b_m - L}_{\varepsilon} \leq b_n - L.$$

Since $n \geq N$ we know

$$b_n - L = |b_n - L| < \varepsilon.$$

since $b_n - L > 0$

Combining we have $\varepsilon \leq b_n - L < \varepsilon$.

Contradiction.

Hence $b_n \leq L$ for all $n \geq 1$.

(2) Similar to the proof for problem 1.

Try it.

3 (a) Let $\varepsilon > 0$.

Since $s_n \to s$ there exists $N_1 > 0$ where

$$\underbrace{|s_n - s| < \varepsilon}_{\text{same as: } s-\varepsilon < s_n < s+\varepsilon} \quad \text{for } n \geq N_1$$

Since $t_n \to t$ there exists $N_2 > 0$ where

$$\underbrace{|t_n - t| < \varepsilon}_{\text{same as: } t-\varepsilon < t_n < t+\varepsilon} \quad \text{for } n \geq N_2$$

Let $N = \max\{N_1, N_2\}$.

So, if $n \geq N$ then both

$$s - \varepsilon < s_n < s + \varepsilon$$

and $t - \varepsilon < t_n < t + \varepsilon$.

We now consider two cases for some $n \geq N$.

$\boxed{\text{Case 1:}}$ Suppose $\max\{s_n, t_n\} = s_n$ for some fixed n with $n \geq N$.

Then from the previous page we have

$$\max\{s_n, t_n\} = s_n < s + \varepsilon \leq \max\{s, t\} + \varepsilon$$

So, $\boxed{\max\{s_n, t_n\} < \max\{s, t\} + \varepsilon}$

Now we get a lower bound on $\max\{s_n, t_n\}$.

We need two sub-cases.

If $\max\{s, t\} = s$, then

$$\max\{s, t\} - \varepsilon = s - \varepsilon < s_n = \max\{s_n, t_n\}$$

So, in this sub-case, $\boxed{\max\{s, t\} - \varepsilon < \max\{s_n, t_n\}}$

If $\max\{s, t\} = t$, then

$$\max\{s, t\} - \varepsilon = t - \varepsilon < t_n \leq \max\{s_n, t_n\}$$

So, in this sub-case, $\boxed{\max\{s, t\} - \varepsilon < \max\{s_n, t_n\}}$

Combining the above we have that

$$\boxed{\max\{s, t\} - \varepsilon < \max\{s_n, t_n\} < \max\{s, t\} + \varepsilon}$$

Case 2: Suppose $\max\{s_n, t_n\} = t_n$ for some fixed n with $n \geq N$.

Then from the previous page we have

$$\max\{s_n, t_n\} = t_n < t + \varepsilon \leq \max\{s, t\} + \varepsilon$$

So, $\boxed{\max\{s_n, t_n\} < \max\{s, t\} + \varepsilon}$

Now we get a lower bound on $\max\{s_n, t_n\}$.

We need two sub-cases.

If $\max\{s, t\} = s$, then

$$\max\{s, t\} - \varepsilon = s - \varepsilon < s_n \leq \max\{s_n, t_n\}$$

So, in this sub-case, $\boxed{\max\{s, t\} - \varepsilon < \max\{s_n, t_n\}}$

If $\max\{s, t\} = t$, then

$$\max\{s, t\} - \varepsilon = t - \varepsilon < t_n = \max\{s_n, t_n\}$$

So, in this sub-case, $\boxed{\max\{s, t\} - \varepsilon < \max\{s_n, t_n\}}$

Combining the above we have that

$$\boxed{\max\{s, t\} - \varepsilon < \max\{s_n, t_n\} < \max\{s, t\} + \varepsilon}$$

Thus, combining cases 1 and 2 we get that if $n \geq N$, then

$$\max\{s,t\} - \varepsilon < \max\{s_n, t_n\} < \max\{s,t\} + \varepsilon$$

Thus, if $n \geq N$, then

$$|\max\{s_n, t_n\} - \max\{s,t\}| < \varepsilon.$$

So,

$$\lim_{n \to \infty} \max\{s_n, t_n\} = \max\{s,t\}.$$

3 (b)

This proof is similar to the proof of 3(a).

Try it.

(4) Suppose $(a_n)_{n=1}^{\infty}$ converges to L.

Let $\varepsilon = 1$.

Then there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < 1$.

Thus, if $n \geq N$ then

$$\begin{aligned} |a_n| &= |a_n - L + L| \\ &\leq |a_n - L| + |L| \\ &< 1 + |L| \end{aligned}$$

Let

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |L| + 1\}.$$

Then, $|a_n| \leq M$ for all $n \geq 1$.