4680 - HW 4

Solutions

① For problem 1 we use this theorem from class:

Given $f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, then

$$\lim_{z \to z_0} f(x,y) = u_0 + i v_0 = w_0$$

iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

1(a) Let $c = c_0 + i c_1$. Then

$$\lim_{z \to z_0} c = \underbrace{\lim_{(x,y)\to(x_0,y_0)} c_0}_{\text{Calc III limits}} + i \underbrace{\lim_{(x,y)\to(x_0,y_0)} c_1}_{\text{Calc III limits}}$$

equal if the RHS exists

$$= c_0 + i c_1 = c$$

Calc III, limit of a constant is a constant

1(b) Let $a = a_0 + i a_1$, $b = b_0 + i b_1$, $z = x + iy$, and $z_0 = x_0 + i y_0$.

$$\lim_{z \to z_0} (az+b) = \lim_{x+iy \to x_0+iy_0} (a_0 + i a_1)(x+iy) + (b_0 + i b_1)$$

$$= \lim_{x+iy \to x_0+iy_0} \left[(a_0 x - a_1 y + b_0) + i(a_1 x + a_0 y + b_1) \right]$$

$$= \lim_{(x,y)\to(x_0,y_0)} (a_0 x - a_1 y + b_0) + i \left[\lim_{(x,y)\to(x_0,y_0)} (a_1 x + a_0 y + b_1) \right]$$

If RHS exists

$$= (a_0 x_0 - a_1 y_0 + b_0) + i(a_1 x_0 + a_0 y_0 + b_1)$$

Calc III limits of continuous functions (polynomials) so we can just plug in x_0, y_0

$$= (a_0 + i a_1)(x_0 + i y_0) + (b_0 + i b_1)$$

$$= a z_0 + b.$$

1(c) Let $c = c_0 + i c_1$, $z = x + iy$,

$z_0 = x_0 + i y_0$.

Then

$$\lim_{z \to z_0} (z^2 + c) = \lim_{x+iy \to x_0 + iy_0} \left((x+iy)^2 + (c_0 + ic_1) \right)$$

$$= \lim_{x+iy \to x_0+iy_0} \left[(x^2 - y^2 + c_0) + i(2xy + c_1) \right]$$

$$= \underbrace{\lim_{(x,y)\to(x_0,y_0)} \left[x^2 - y^2 + c_0 \right]} + i \underbrace{\lim_{(x,y)\to(x_0,y_0)} \left[2xy + c_1 \right]}$$

If RHS exists

Calc III
limits of continuous (polynomials)
functions
so can plug in x_0, y_0

$$= \left[x_0^2 - y_0^2 + c_0 \right] + i \left[2 x_0 y_0 \right)$$

$$= (x_0 + i y_0)^2 + (c_0 + i c_1)$$

$$= z_0^2 + c$$

1(d) Let $z = x+iy$, $z_0 = x_0+iy_0$

Then,

$$\lim_{x+iy \to x_0+iy_0} \mathrm{Re}(z) = \lim_{x+iy \to x_0+iy_0} \left[x + i0 \right]$$

$$= \lim_{(x,y)\to(x_0,y_0)} x + i \lim_{(x,y)\to(x_0,y_0)} 0$$

$$= x_0 + i\,0 = \mathrm{Re}(z_0)$$

② Let $z = x + iy$ and $z_0 = x_0 + iy_0$.

Using the same theorem that we used in problem 1 we get:

$$\lim_{z \to z_0} \overline{z} = \lim_{x+iy \to x_0+iy_0} \overline{(x+iy)}$$

$$= \lim_{x+iy \to x_0+iy_0} x - iy = \underbrace{\lim_{(x,y)\to(x_0,y_0)} x}_{} + i \underbrace{\lim_{(x,y)\to(x_0,y_0)} (-y)}_{}$$

Calc III limits of continuous functions so can plug x_0, y_0 into them

Change to calculus III limits

$$= x_0 + i(-y_0) = x_0 - iy_0$$

$$= \overline{z_0}$$

Thus, $\overline{z}$ is continuous at all $z_0 \in \mathbb{C}$, since $\lim_{z \to z_0} \overline{z} = \overline{z_0}$.

$\boxed{3}$ Same tactic as problem 2.

Let $z = x + iy$ and $z_0 = x_0 + iy_0$

Then,

$$\lim_{z \to z_0} |z| = \lim_{x+iy \to x_0+iy_0} \left[\sqrt{x^2+y^2} + i0 \right]$$

$$= \underbrace{\lim_{(x,y)\to(x_0,y_0)} \sqrt{x^2+y^2}} + i \underbrace{\lim_{(x,y)\to(x_0,y_0)} 0}$$

Calc III limits of continuous functions plug in x_0, y_0

$$= \sqrt{x_0^2+y_0^2} + i0$$

$$= |z_0|.$$

Since $\lim_{z\to z_0} |z| = |z_0|$ for all $z_0 \in \mathbb{C}$, $|z|$ is continuous on all of $\mathbb{C}$.

④ Suppose the given conditions of the problem and

$\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} f(z) = L_2$.

Let $\varepsilon > 0$.

Since $\lim_{z \to z_0} f(z) = L_1$, there exists a $\delta_1 > 0$ so that if $z \in A$ and $\underbrace{0 < |z - z_0| < \delta_1}_{\text{z is } \delta \text{ close to } z_0 \text{ but } z \neq z_0}$

then $|f(z) - L_1| < \varepsilon/2$.

Similarly there exists $\delta_2 > 0$ so that if $z \in A$ and $0 < |z - z_0| < \delta$ then $|f(z) - L_2| < \varepsilon/2$,

Thus, if $z \in A$ and

$0 < |z - z_0| < \underbrace{\min\{\delta_1, \delta_2\}}_{\text{this means the smaller of } \delta_1 \text{ or } \delta_2.}$

then

$$|L_1 - L_2| = |L_1 - f(z) + f(z) - L_2|$$
$$\leq |L_1 - f(z)| + |f(z) - L_2|$$
$$= |f(z) - L_1| + |f(z) - L_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $|L_1 - L_2| < \varepsilon$
for all positive ε.

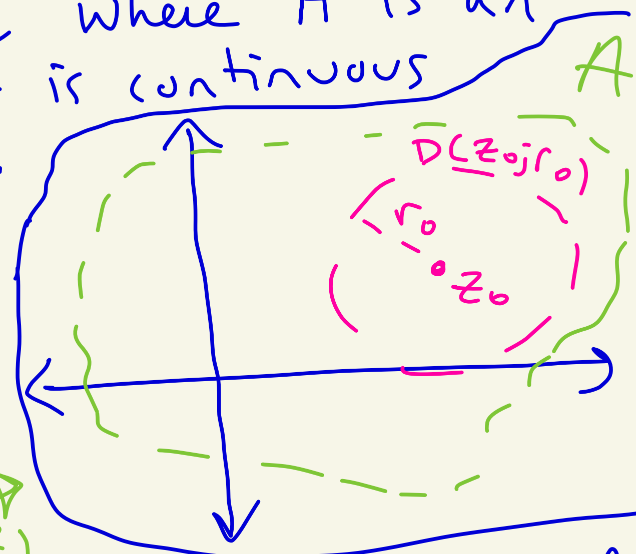
Thus, $|L_1 - L_2| = 0$.

So, $L_1 - L_2 = 0$

Thus, $L_1 = L_2$. $\blacksquare$

(5) Suppose that $f: A \to \mathbb{C}$ where A is an open set. Suppose that f is continuous at $z_0 \in A$ and that $f(z_0) \neq 0$.
and that $f(z_0) \neq 0$

First note that since $z_0 \in A$ and A is open, there exists $r_0 > 0$ so that $D(z_0; r_0) \subseteq A$. (see pic)

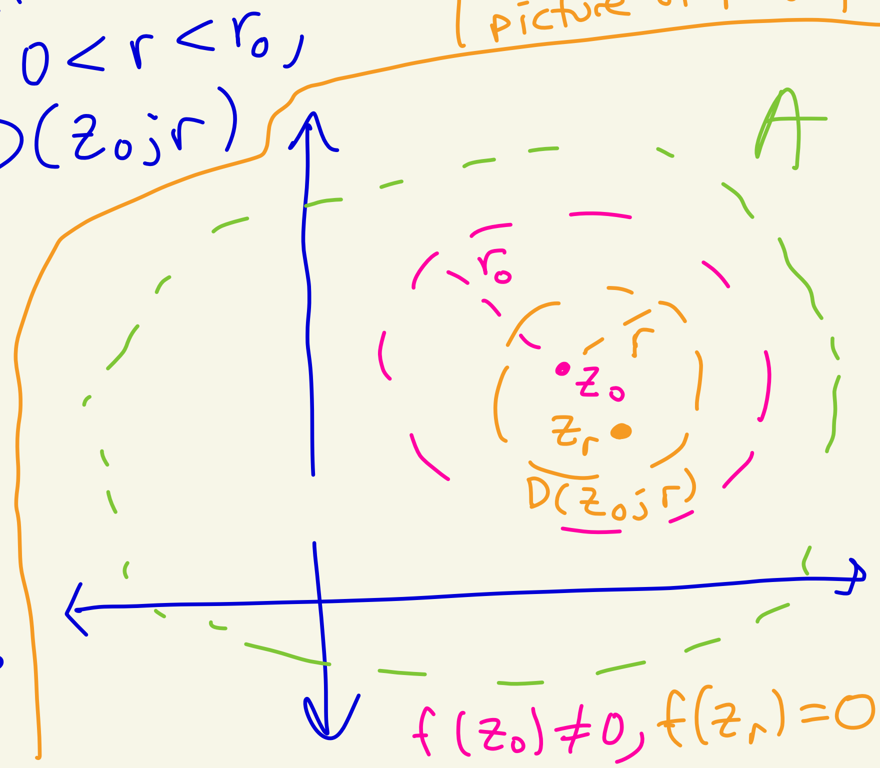


We want to show exists an r-neighborhood $D(z_0; r) \subseteq A$ such that $f(z) \neq 0$ for all $z \in D(z_0; r)$. We may only consider $r < r_0$ since we can always shrink down.

We show this by contradiction.

That is suppose that no such neighborhood exists. That is, for every r with $0 < r < r_0$, there exists $z_r \in D(z_0; r)$ with $f(z_r) = 0$.

picture of proof



We show this contradicts the fact that f is continuous at z_0

Let $L = f(z_0) \neq 0$.

Since f is continuous at z_0 this means that given $\varepsilon = \frac{|L|}{2} > 0$

there exists a $0 < \delta < r_0$ such that if $z \in A$ and $|z - z_0| < \delta$

we may assume $\delta < r_0$ by shrinking it if it isn't

then

$$|f(z) - \underbrace{L}_{f(z_0)}| < \underbrace{\frac{|L|}{2}}_{\varepsilon}.$$

since $\lim_{z \to z_0} f(z) = f(z_0) = L$ since f is continuous at z_0

But from the previous page there exists $\underbrace{z_\delta \in D(z_0; \delta)}_{\text{ie } |z_\delta - z_0| < \delta}$

with $f(z_\delta) = 0$. But then,

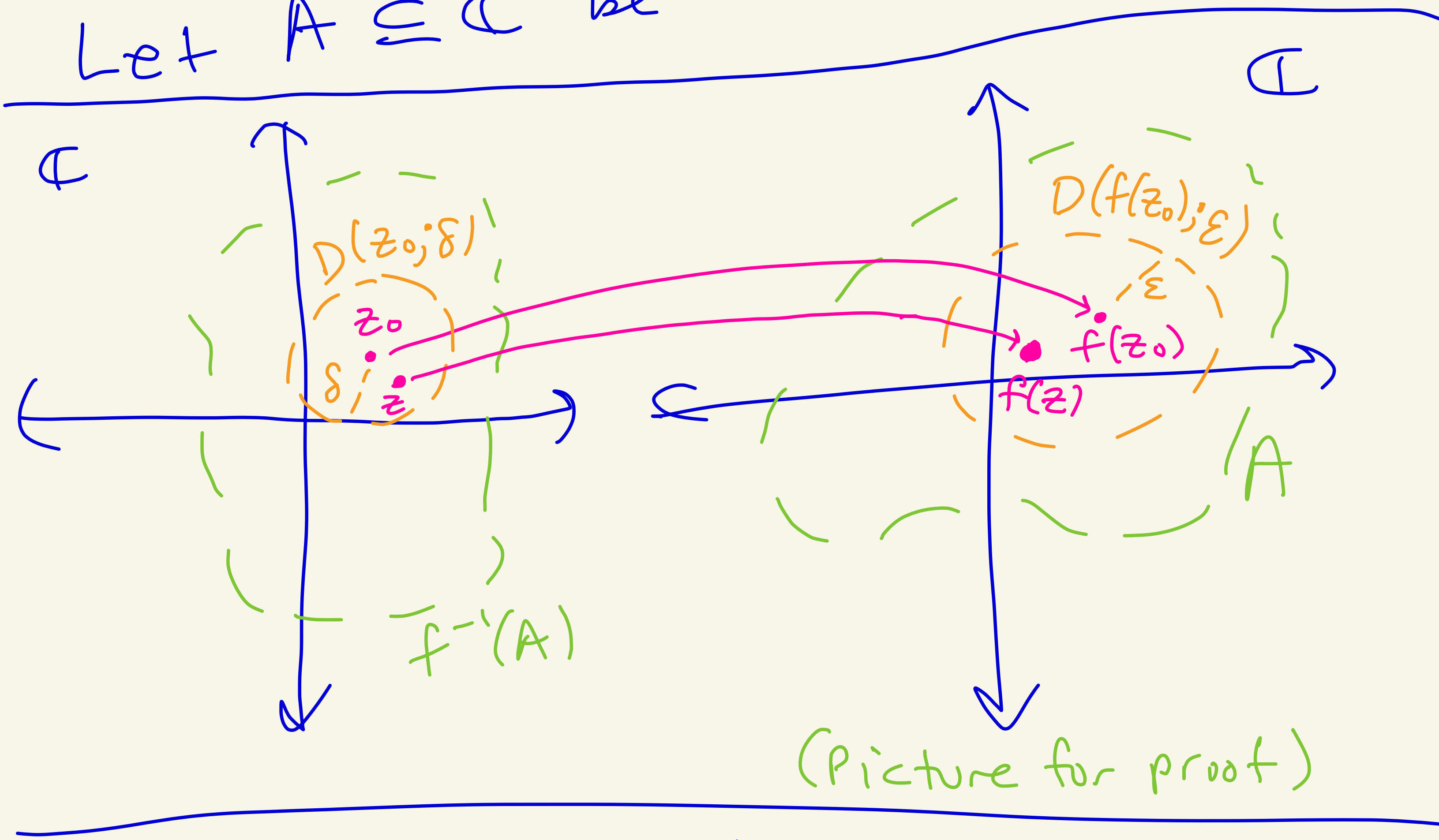
$$|L| = |0 - L| = |f(z_\delta) - L| < \frac{|L|}{2}$$

So, $|L| < \frac{|L|}{2}$. But this can't happen since $L \neq 0$. Contradiction. $\boxtimes$

⑥ Let $f : \mathbb{C} \to \mathbb{C}$.

($\Longrightarrow$) Suppose that f is continuous on all of $\mathbb{C}$.

Let $A \subseteq \mathbb{C}$ be an open set.



We want to show that

$$f^{-1}(A) = \{ z \in \mathbb{C} \mid f(z) \in A \}$$

is open.

Let $z_0 \in f^{-1}(A)$. Then $f(z_0) \in A$.

Since A is open and $f(z_0) \in A$, there exists $\varepsilon > 0$ so that $D(f(z_0); \varepsilon) \subseteq A$.

That is, if $|w - f(z_0)| < \varepsilon$ then $w \in A$.

Since f is continuous at z_0, given $\varepsilon > 0$ there exists $\delta > 0$ so that if $z \in \mathbb{C}$ and $\underbrace{|z - z_0| < \delta \text{ then } |f(z) - f(z_0)| < \varepsilon}_{\text{since } \lim_{z \to z_0} f(z) = f(z_0)}$.

That means for all $z \in \mathbb{C}$ with $|z - z_0| < \delta$ then $f(z) \in D(f(z_0); \varepsilon)$.

So if $z \in D(z_0; \delta)$, then $f(z) \in D(f(z_0); \varepsilon) \subseteq A$.

So, if $z \in D(z_0; \delta)$, then $\underbrace{z \in f^{-1}(A)}_{\text{since } f(z) \in A}$

Summarizing, if $z_0 \in f^{-1}(A)$, there exists a δ-neighborhood of z_0 contained in $f^{-1}(A)$.

So, $f^{-1}(A)$ is open.

(($\Leftarrow$) on the next page)

($\Leftarrow$) Suppose $f^{-1}(A)$ is open for every open set $A \subseteq \mathbb{C}$.

Let $z_0 \in \mathbb{C}$.
Let's show $\lim_{z \to z_0} f(z) = f(z_0)$.
Let $\varepsilon > 0$.
Let $A = D(f(z_0); \varepsilon)$.
Since A is open (HW 3), by the assumption above, $f^{-1}(A)$ is open.
We have that $z_0 \in f^{-1}(A)$ since $f(z_0) \in A$.
Since $f^{-1}(A)$ is open there exists $\delta > 0$ so that $D(z_0; \delta) \subseteq f^{-1}(A)$.
That is, if $z \in D(z_0; \delta)$ then $z \in f^{-1}(A)$.
That is, if $|z - z_0| < \delta$ then $f(z) \in A$.
That is, if $|z - z_0| < \delta$ then $\underbrace{|f(z) - f(z_0)| < \varepsilon}_{f(z) \in A = D(f(z_0);\varepsilon)}$
Thus, $\lim_{z \to z_0} f(z) = f(z_0)$. ▨

Picture for proof