# 4680 - HW 5

Solutions

$\boxed{1}$(a) $f(z) = 13z^7 - 3z^4 + 1$

is a polynomial, so by class, is analytic on all of $\mathbb{C}$. So, $f$ is entire.

$$f'(z) = 13 \cdot 7 z^6 - 3 \cdot 4 z^3$$
$$= 91z^6 - 12z^3 \quad \text{for all } z \in \mathbb{C}$$

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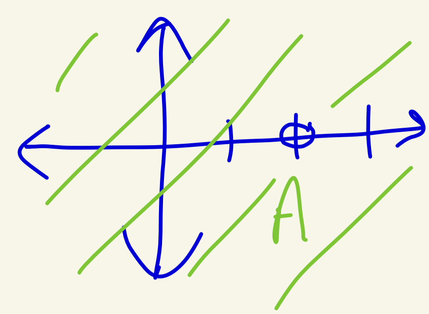
$\boxed{1}$(b) $f(z) = \dfrac{3z^2 - 1}{2 - z}$ is a rational function so is analytic where the denominator is not zero.

$2 - z = 0$ when $z = 2$.

So, $f$ is analytic on $A = \mathbb{C} - \{2\}$

$$f'(z) = \frac{6z(2-z) - (3z^2-1)(-1)}{(2-z)^2}$$




$$= \frac{12z - 6z^2 + 3z^2 - 1}{(2-z)^2} = \frac{-3z^2 + 12z - 1}{(2-z)^2}, \ \forall z \in A$$

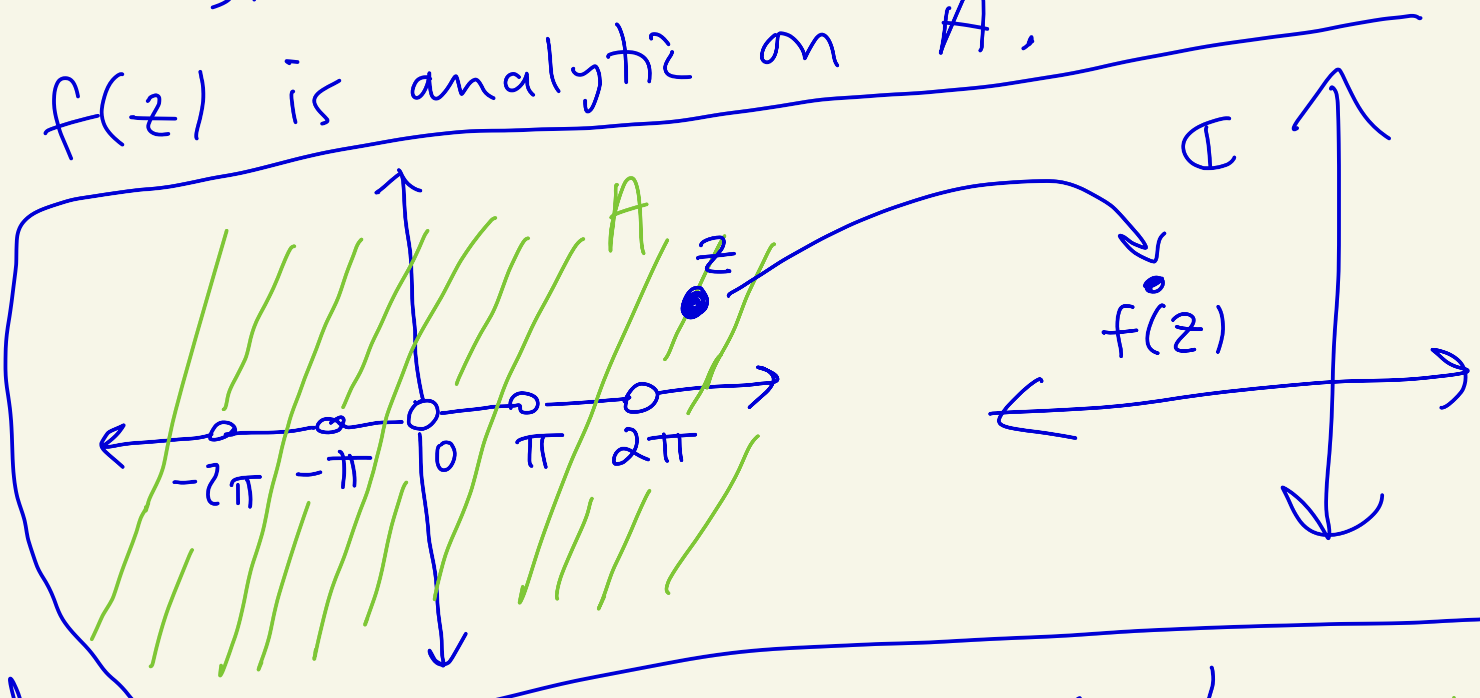
1(c) $f(z) = \dfrac{\cos(z)}{\sin(z)}$

From HW 2, $\sin(z) = 0$ iff $z = n\pi$, $n \in \mathbb{Z}$.

Let $A = \mathbb{C} - \{\pi n \mid n \in \mathbb{Z}\}$.

From class, $\sin(z)$ and $\cos(z)$ are analytic on all of $\mathbb{C}$. Thus, $f(z) = \dfrac{\cos(z)}{\sin(z)}$ is analytic where $\sin(z) \neq 0$.

So, $f(z)$ is analytic on $A$.




And,

$$f'(z) = \frac{(\cos(z))'\sin(z) - \cos(z)(\sin(z))'}{\sin^2(z)}$$

$$= \frac{-\sin(z)\sin(z) - \cos(z)\cos(z)}{\sin^2(z)} = \frac{-\overbrace{(\sin^2(z) + \cos^2(z))}^{=1 \text{ by HW 2}}}{\sin^2(z)}$$

$$= -\frac{1}{\sin^2(z)} \quad \forall z \in A$$

$\boxed{1}$(d) $f(z) = \left(\dfrac{1}{z-1}\right)^{100} = (z-1)^{-100}$

Let $g(z) = z^{100}$ and $h(z) = \dfrac{1}{z-1}$.

Then $f(z) = (g \circ h)(z) = g(h(z))$.

$g(z)$ is entire, it is analytic on all of $\mathbb{C}$.

$h(z)$ is analytic on $A = \mathbb{C} - \{1\}$

The composition $f(z)$ is analytic on $A$.

with $f'(z) = -100(z-1)^{-101} = \dfrac{-100}{(z-1)^{101}}$ for all $z \in A$

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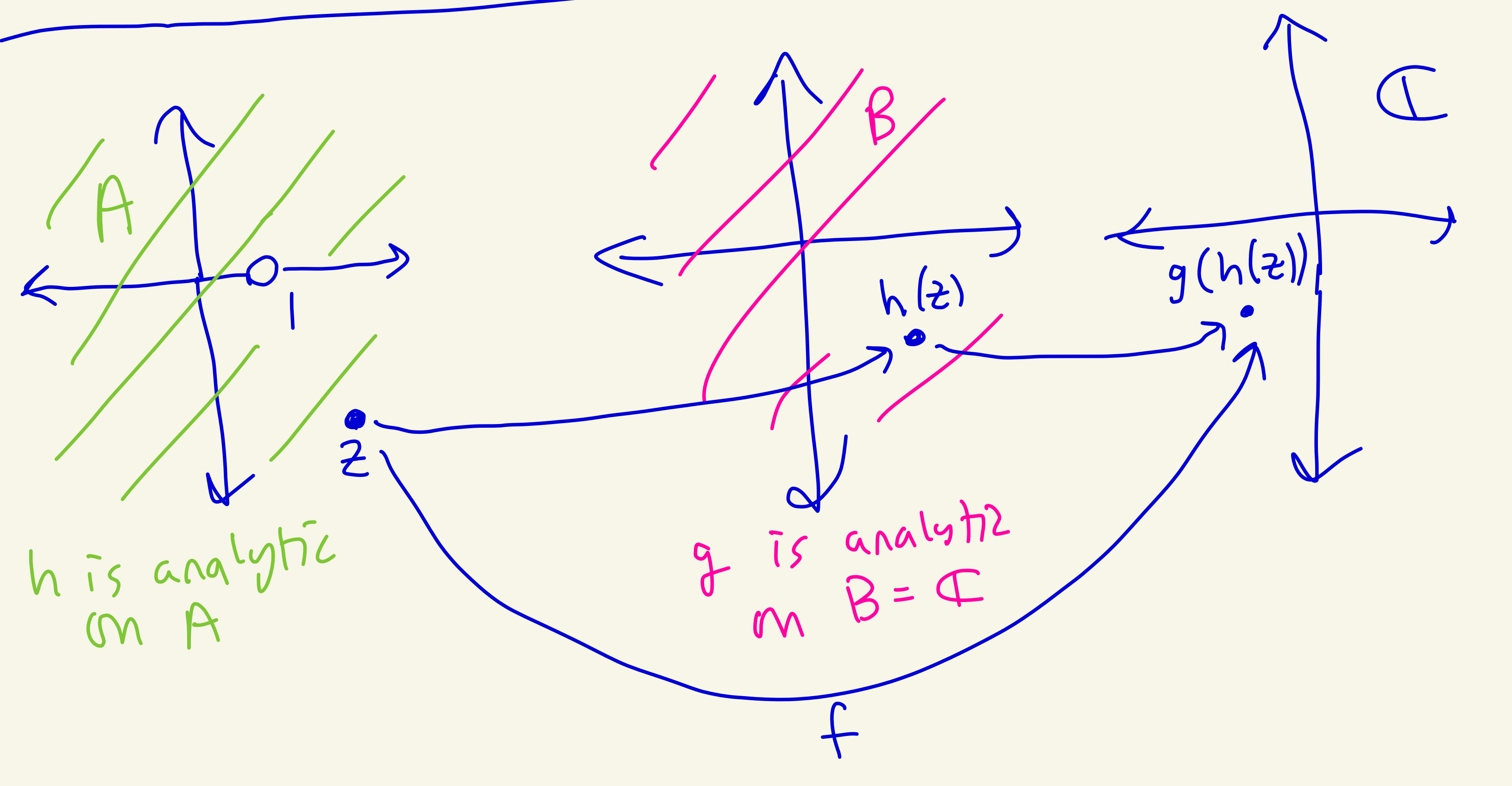
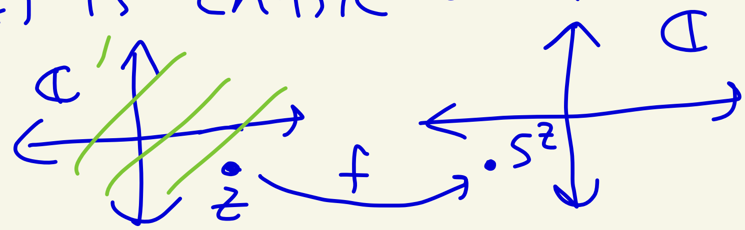


1(e) Let $f(z) = 5^z = e^{z\log(5)}$ using the principal branch of the logarithm. From class, $f(z)$ is entire and $f'(z) = \log(5) \cdot 5^z$.

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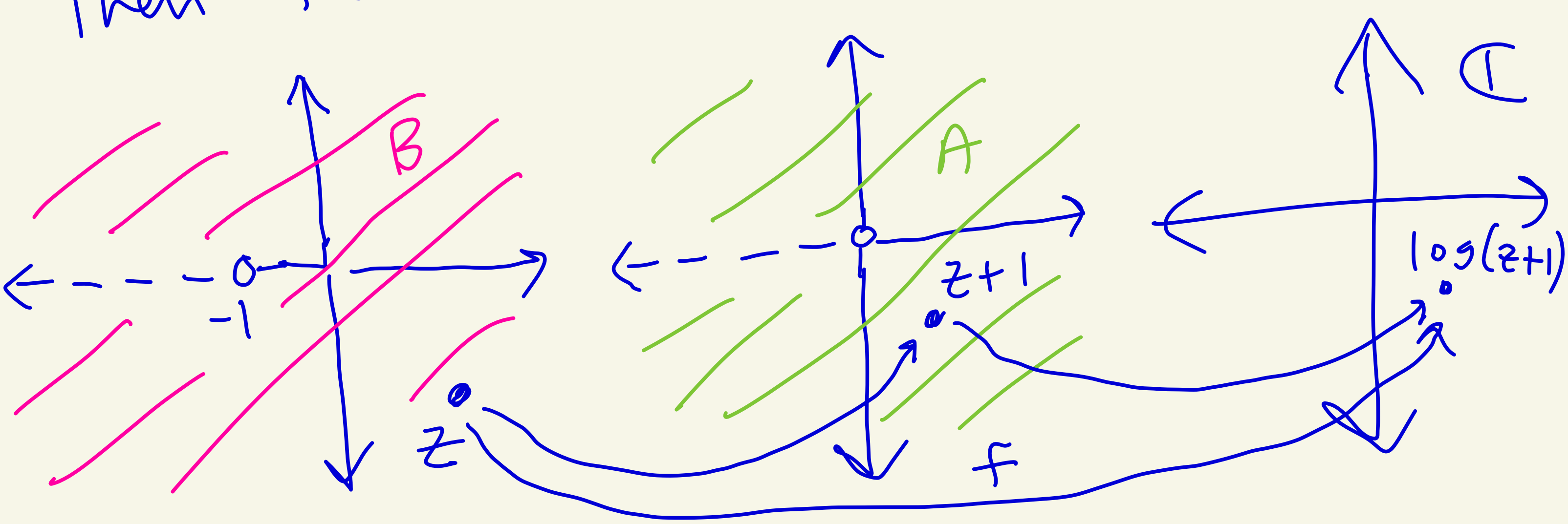
1(f) From class, the principal branch of the logarithm $\log(w)$ is analytic on $A = \mathbb{C} - \{x+iy \mid x \le 0,\ y = 0\}$

When is $z+1 \notin A$?

Let $z = x+iy$.
Then $z+1 = (x+1)+iy$. Then $z+1 \notin A$ iff $x+1 \le 0,\ y=0$ iff $x \le -1,\ y = 0$.

Let $B = \mathbb{C} - \{x+iy \mid x \le -1,\ y = 0\}$.
Then $f(z) = \log(z+1)$ is analytic on $B$.



and $f'(z) = \left(\log(z+1)\right)'$

$$= \frac{1}{z+1} \cdot 1 = \frac{1}{z+1} \quad \forall z \in B$$

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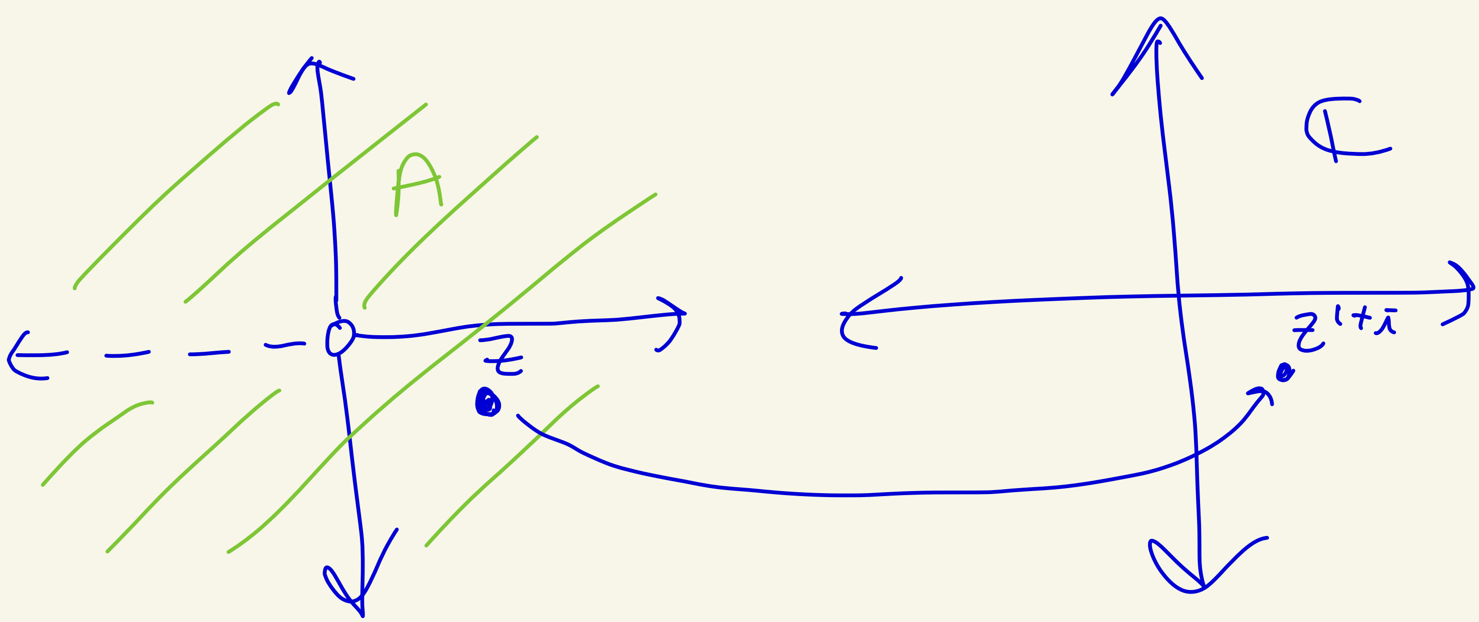
$\boxed{1}$ (g) Let $f(z) = z^{1+i} = e^{(1+i)\log(z)}$, where $\log(z)$ is defined as the principal branch of the log

Let $A = \mathbb{C} - \{x+iy \mid x \le 0, y = 0\}$

From class, $f$ is analytic on $A$

and $f'(z) = (1+i)z^{(1+i)-1}$

$$= (1+i)z^{i} \quad \forall z \in A$$



$\boxed{1}$ (h) $f(z) = (z-2)^{1/2} = h(g(z))$

Let $h(z) = z^{1/2} = e^{1/2 \log(z)}$ defined using the brincipal branch of the logarithm and $g(z) = z-2$.

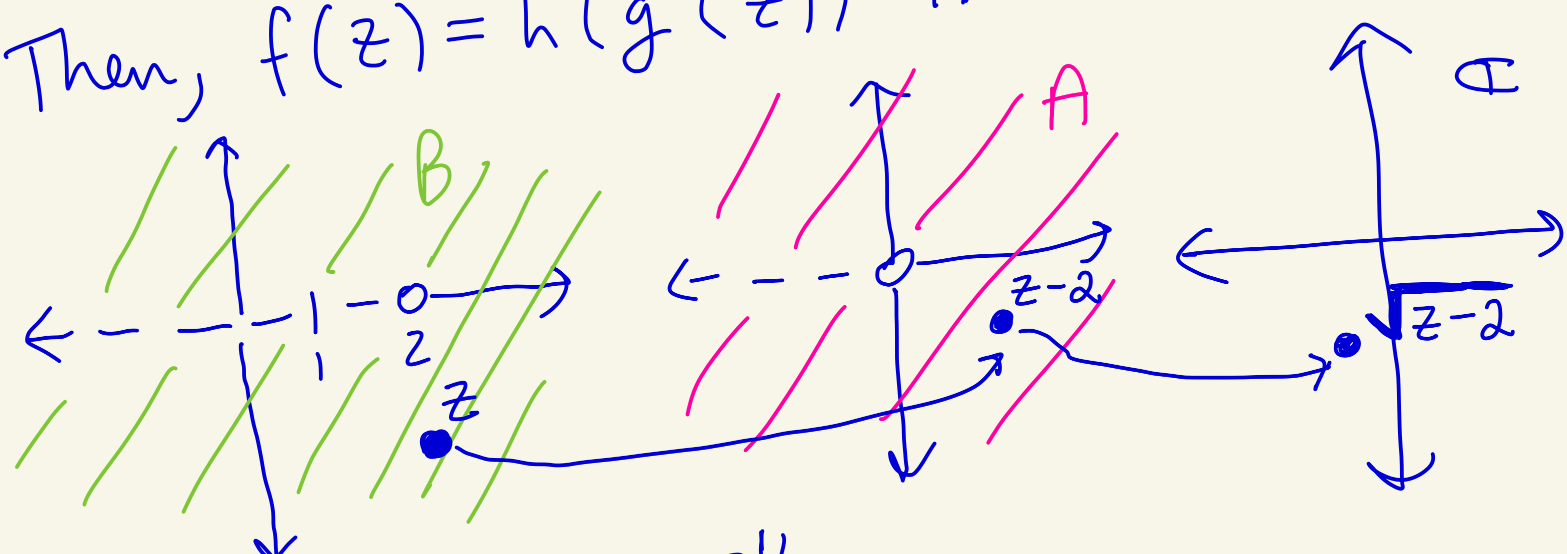
Let $A = \mathbb{C} - \{x+iy \mid x \le 0, y=0\}$

Then, $h(z)$ is analytic on $A$, $g(z) = z-2$ is analytic on all of $\mathbb{C}$.

Let $z = x+iy$. Then $z-2 \notin A$ iff $x-2 \le 0, y=0$ iff $(x-2)+iy \notin A$ iff $x \le 2, y=0$.

Set $B = \mathbb{C} - \{x+iy \mid x \le 2, y=0\}$.

Then, $f(z) = h(g(z))$ is analytic on $B$.




And $f'(z) = \frac{1}{2}(z-2)^{-1/2}$

$\boxed{2}$ (a) $f(z) = |z|$

$$f(x+iy) = \underbrace{\sqrt{x^2+y^2}}_{u(x,y)} + i\underbrace{0}_{v(x,y)}$$

Cauchy-Riemann time:

$$\frac{\partial u}{\partial x}(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = 0$$

Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

these become

$$(*) \ \underbrace{\frac{x}{\sqrt{x^2+y^2}} = 0}_{x=0,\, y\neq 0}, \quad (**) \ \underbrace{\frac{y}{\sqrt{x^2+y^2}} = 0}_{y=0,\, x\neq 0}$$

$(*)$ is solved when $x=0$ & $y \neq 0$.
$(**)$ is solved when $y=0$ & $x \neq 0$.
There are no common solutions to $(*)$ and $(**)$.
Thus, $f(z) = |z|$ is analytic nowhere.

$\boxed{2}$ (b) $f(z) = e^{\bar{z}}$

Let $z = x + iy$.

Then $f(z) = e^{\bar{z}} = e^{x - iy} = e^x e^{i(-y)}$

$= e^x \left[\underbrace{\cos(-y)}_{\cos(y)} + i \underbrace{\sin(-y)}_{-\sin(y)}\right]$

$= \underbrace{e^x \cos(y)}_{u(x,y)} + i \left[\underbrace{-e^x \sin(y)}_{v(x,y)}\right]$

$u(x,y) = e^x \cos(y)$

$v(x,y) = -e^x \sin(y)$

Cauchy-Riemann: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = e^x \cos(y)$

$\frac{\partial v}{\partial y} = -e^x \cos(y)$

(*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ iff $e^x \cos(y) = -e^x \cos(y)$

iff $2e^x \cos(y) = 0$

iff $\cos(y) = 0$

iff $y \in \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\}$

$x \in \mathbb{R}$

$\frac{\partial u}{\partial y} = -e^x \sin(y)$

$-\frac{\partial v}{\partial x} = -[-e^x \sin(y)]$

$= e^x \sin(y)$

(**) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ iff $-e^x \sin(y) = e^x \sin(y)$

iff $2e^x \sin(y) = 0$

iff $y \in \{0, \pm\pi, \pm 2\pi, \ldots\}$

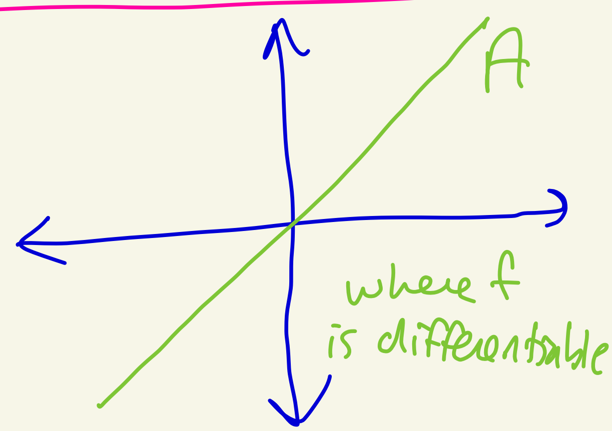
$x \in \mathbb{R}$

There are no common solutions to (*) and (**) so $f(z)$ is not analytic anywhere

$\boxed{3}$ (a) $f(x+iy) = \underbrace{x^2}_{u(x,y)=x^2} + i\underbrace{y^2}_{v(x,y)=y^2}$

$\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 2y$ (*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ iff $2x = 2y$ iff $x = y$

$\dfrac{\partial u}{\partial y} = 0$, $-\dfrac{\partial v}{\partial x} = 0$ (**) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ for all $x, y$.




The common solutions to (*) and (**) is the set $A = \{x + iy \mid x = y\}$
$= \{a + ia \mid a \in \mathbb{R}\}$.

Note that $u(x,y) = x^2$ and $v(x,y) = y^2$ are continuous for all $(x,y)$ with $x = y$.

Thus, $f(x+iy) = x^2 + iy^2$ is differentiable on A with derivative

$$f'(x+iy) = \underbrace{2x}_{\partial u/\partial x} + i\underbrace{0}_{\partial v/\partial x} = 2x$$

Note f is not analytic on A. To be analytic at a point you must be differentiable on an open set containing that point.

$\boxed{3}$ (b) $f(z) = z \cdot \mathrm{Im}(z)$

$$f(x+iy) = (x+iy)\cdot y = \underbrace{xy}_{u(x,y)=xy} + i\,\underbrace{y^2}_{v(x,y)=y^2}$$

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$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= y \\ \frac{\partial v}{\partial y} &= 2y \end{aligned}\right\} \quad (*) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ iff } y = 2y \text{ iff } y=0,\ x \in \mathbb{R}$$

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$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= x \\ -\frac{\partial v}{\partial x} &= 0 \end{aligned}\right\} \quad (**) \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ iff } x=0,\ y \in \mathbb{R}$$

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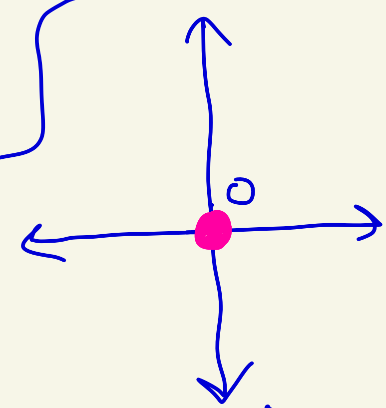
$(*)$ and $(**)$ have common solution $x+iy = 0 + i0 = 0$.

So, $f$ is differentiable only at $z = 0$ with

$$f(x+iy) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0 + i0 = 0$$




where $f$ has a derivative

[Note that $f$ is not analytic at 0 since $f$ is not differentiable in a neighborhood of 0]

$\boxed{4}$ $f'(0)$ if it existed would be equal to

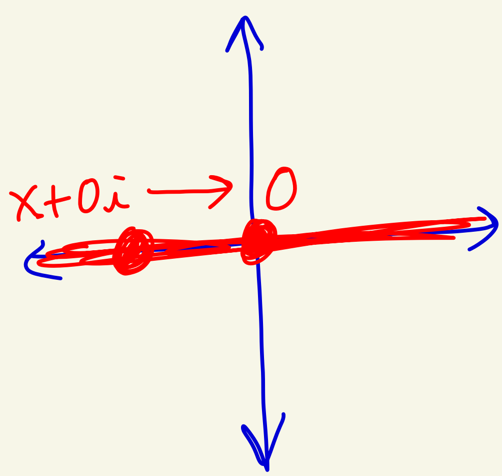
$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\frac{(\overline{z})^2}{z} - 0}{z-0}$$

$$= \lim_{z\to 0} \left(\frac{\overline{z}}{z}\right)^2 = \lim_{x+iy\to 0} \left(\frac{x-iy}{x+iy}\right)^2$$

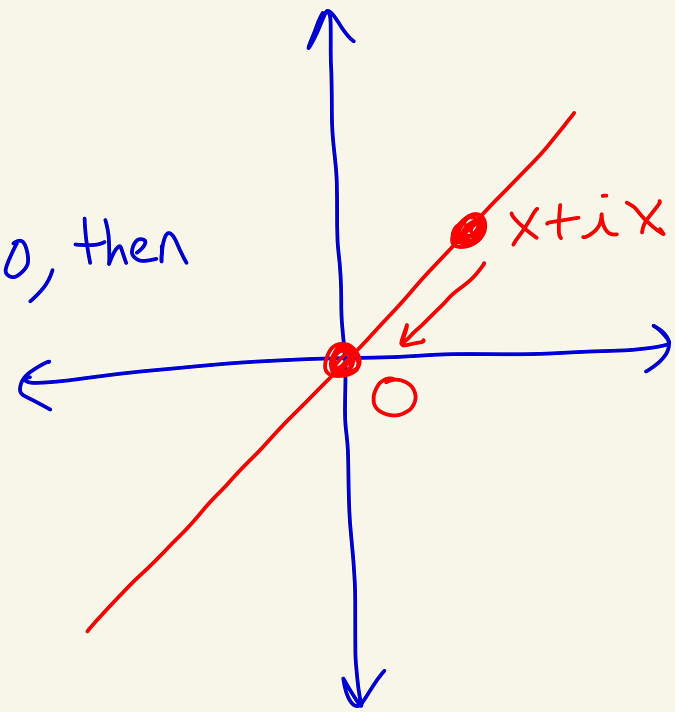
If the limit exists, then it wouldn't matter what direction we approached 0 from, we would get the same answer.

Suppose $x+iy \to 0+i0$ along the x-axis. Ie suppose $y=0$, and $x \neq 0$. Then,

$$\left(\frac{x-iy}{x+iy}\right)^2 = \left(\frac{x}{x}\right)^2 = 1$$




So, approaching 0 along the x-axis we get 1, Now let's appoach 0 along the line $y = x$.




Suppose $y = x$ and $y \neq 0, x \neq 0$, then

$$\left(\frac{x - iy}{x + iy}\right)^2 = \left(\frac{x - ix}{x + ix}\right)^2$$

$$= \frac{x^2 - 2ix^2 - x^2}{x^2 + 2ix^2 - x^2} = -1$$

So as $x + iy \to 0$ along the line $y = x$ we get $-1$.

Since we get $1$ approaching $0$ on the $x$-axis and $-1$ approaching $0$ on the line $y = x$, the limit $\lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z - 0}$ does not exist. Hence, $f'(0)$ does not exist.

$\boxed{5}$ Let $g: A \to \mathbb{C}$ be analytic on $A$ where $A \subseteq \mathbb{C}$ is an open set.

Let $B = \{ z \in A \mid g(z) \neq 0 \}$.

(i) Let's show that $B$ is open.

Let $z_0 \in B$.

Let's show that $z_0$ is an interior point of $B$.

Since $g$ is analytic on $A$, $g$ is continuous on $A$ [class Thm].

Thus, $g$ is continuous on the open set $A$ containing $z_0$ and $g(z_0) \neq 0$.

Therefore, by HW 4, there exists $r > 0$ so that $g(z) \neq 0$ for all $z \in D(z_0; r)$ with $D(z_0; r) \subseteq A$.

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Since $g(z) \neq 0$ $\forall z \in D(z_0; r)$, and $D(z_0; r) \subseteq A$, we have that $D(z_0; r) \subseteq B$.

So, $z_0$ is an interior point of $B$.

Thus, $B$ is open.

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(ii) Since $1$ is analytic on $B$, and $g(z)$ is analytic on $B$, and $g(z) \neq 0$ on $B$, by thm from class $\frac{1}{g(z)}$ is analytic on $B$. $\blacksquare$