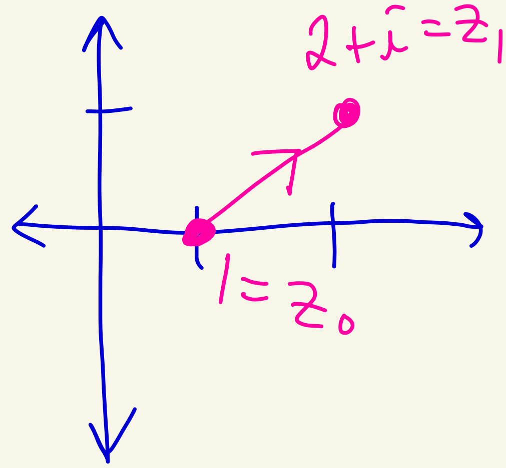
4680 - HW 6

Solutions

1 (a) γ is the line segment joining 1 to $2+i$.

Formula:

$\gamma(t) = z_0 + t(z_1 - z_0)$
$0 \le t \le 1$



We have

$\gamma(t) = 1 + t((2+i) - 1)$

$\gamma(t) = 1 + t(1+i)$

$0 \le t \le 1$

$\gamma'(t) = (1+i)$

$1 + 2t(1+i) + t^2(1+i)^2$
$= 1 + 2t + 2it + t^2(1+2i-1)$
$= 1 + 2t + 2it + 2it^2$

$$\int_\gamma (z^2+2)\,dz = \int_0^1 \underbrace{\left(\left[1+t(1+i)\right]^2 + 2\right)}_{f(\gamma(t))} \underbrace{(1+i)\,dt}_{\gamma'(t)\,dt}$$

$$= \int_0^1 \underbrace{\left[3 + 2t + 2it + 2it^2\right]\left[(1+i)\right]}\,dt$$

$= 3 + 2t + 2it + 2it^2 + 3i + 2it - 2t - 2t^2$
$= (3 - 2t^2) + i(3 + 4t + 2t^2)$

$$= \int_0^1 (3-2t^2)\,dt + i\int_0^1 (3+4t+2t^2)\,dt$$

$$= \left(3t - \frac{2}{3}t^3\right)\Big|_0^1 + i\left(3t + \frac{4t^2}{2} + \frac{2t^3}{3}\right)\Big|_0^1$$

$$= \left(3 - \frac{2}{3}\right) + i\left(3 + 2 + \frac{2}{3}\right)$$

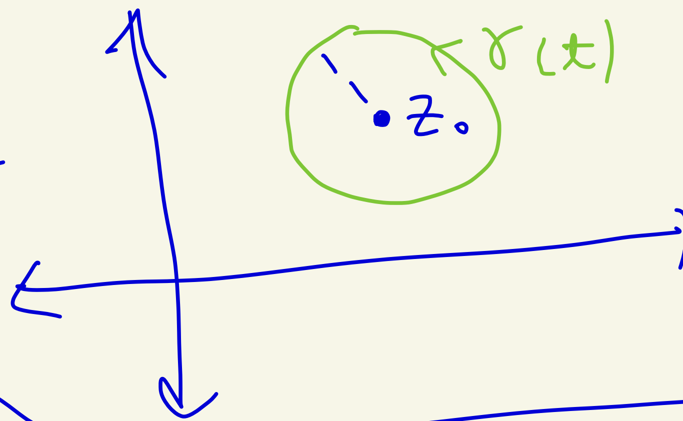
$$= \left(\frac{9}{3} - \frac{2}{3}\right) + i\left(\frac{15}{3} + \frac{2}{3}\right)$$

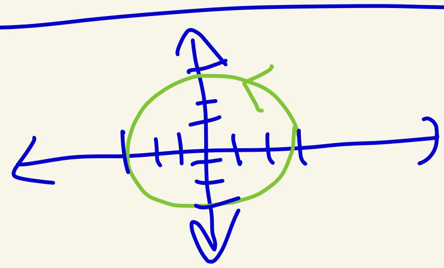
$$= \boxed{\frac{7}{3} + i\,\frac{17}{3}}$$

1 (b)

formula for Circle
center $= z_0$
radius $= r$
once around the circle
going counterclockwise

$\gamma(t) = z_0 + re^{it}$
$0 \le t \le 2\pi$





Our problem has $z_0 = 0$ and $r = 3$

$\gamma(t) = 0 + 3e^{it} = 3e^{it}$, $0 \le t \le 2\pi$

$$\int_\gamma (z^3+z)\,dz = \int_0^{2\pi} \underbrace{\left[(3e^{it})^3 + 3e^{it}\right]}_{f(\gamma(t))} \underbrace{3ie^{it}\,dt}_{\gamma'(t)\,dt}$$

$$= \int_0^{2\pi} \left[27e^{3it} \cdot 3ie^{it} + 3e^{it} \cdot 3ie^{it}\right] dt$$

$$= \int_0^{2\pi} \left[81ie^{4it} + 9ie^{2it}\right] dt$$

$$= \int_0^{2\pi} \left[81i\cos(4t) + \underbrace{i^2 81}_{-81} \sin(4t) + 9i\left[\cos(2t) + i\sin(2t)\right]\right] dt$$

$$= \int_0^{2\pi} \left(-81\sin(4t) - 9\sin(2t)\right) dt$$

$$+ i \int_0^{2\pi} \left(81\cos(4t) + 9\cos(2t)\right) dt$$

$$= \left[\frac{81}{4}\cos(4t) + \frac{9}{2}\cos(2t)\right]\Bigg|_0^{2\pi}$$

$$+ i\left[\frac{81}{4}\sin(4t) + \frac{9}{2}\sin(2t)\right]\Bigg|_0^{2\pi}$$

$$= \left[\frac{81}{4}\underbrace{\cos(8\pi)}_{1} + \frac{9}{2}\underbrace{\cos(4\pi)}_{1} - \frac{81}{4}\underbrace{\cos(0)}_{1} - \frac{9}{2}\underbrace{\cos(0)}_{1}\right]$$

$$+ i\left[\frac{81}{4}\underbrace{\sin(8\pi)}_{0} + \frac{9}{2}\underbrace{\sin(4\pi)}_{0} - \frac{81}{4}\underbrace{\sin(0)}_{0} - \frac{9}{2}\underbrace{\sin(0)}_{0}\right]$$

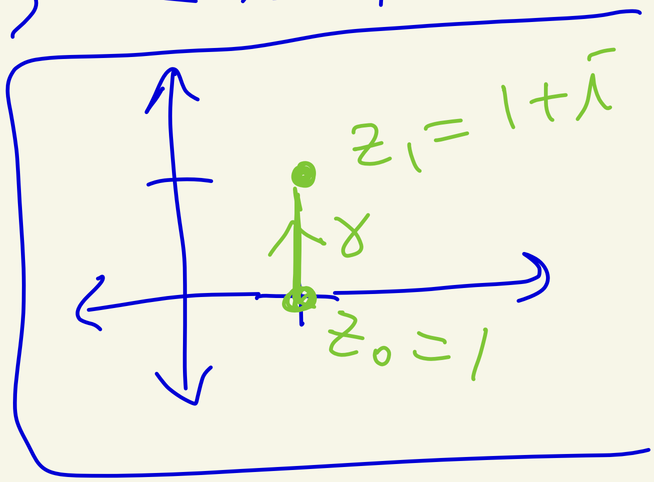
$$= 0$$

$\boxed{1}$ (c) $\gamma(t) = z_0 + t(z_1 - z_0), \quad 0 \le t \le 1$

$$\gamma(t) = 1 + t((1+i) - 1), \quad 0 \le t \le 1$$

$$\gamma(t) = 1 + it, \quad 0 \le t \le 1$$

$$\gamma'(t) = i$$



$$f(x+iy) = x - y$$

$$\int_\gamma (x-y)\,dz$$

$$= \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^1 f(\underset{x=1}{1} + i\underset{y=t}{t}) \cdot i\,dt$$

$$= \int_0^1 (\underbrace{1 - t}_{x-y})\, i\,dt$$

$$= i\int_0^1 (1-t)\,dt = i\left[t - \frac{t^2}{2}\right]_0^1$$

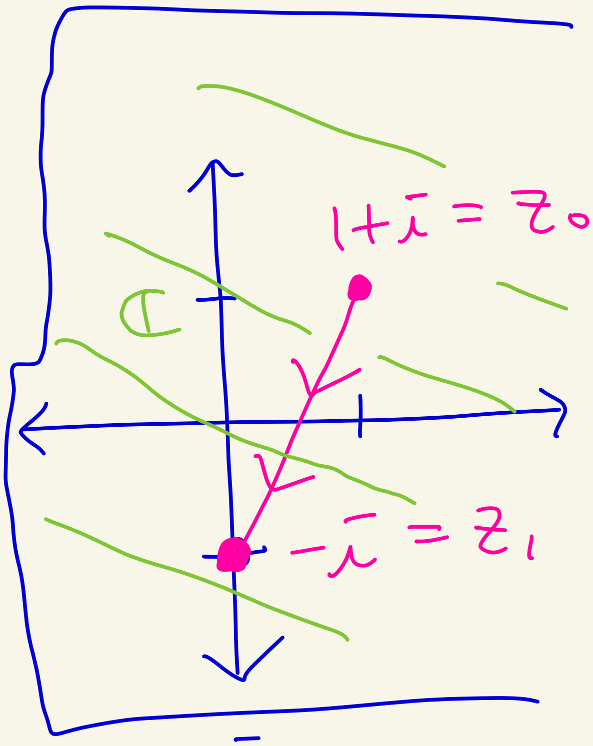
$$= i\left[1 - \frac{1}{2}\right] = \frac{i}{2}$$

② (a) γ is the line segment joining $1+i$ to $-i$.

Formula:

$$\gamma(t) = z_0 + t(z_1 - z_0)$$
$$0 \le t \le 1$$

We have



$$\gamma(t) = (1+i) + t(-i-(1+i))$$

$$\gamma(t) = (1+i) + t(-1-2i)$$
$$0 \le t \le 1$$

$$\int_\gamma \underbrace{\cos(2z)}_{\text{continuous on open set } \mathbb{C} \text{ containing } \gamma} dz = \underbrace{\tfrac{1}{2}\sin(2z)}_{\text{analytic on open set } \mathbb{C} \text{ containing } \gamma} \Big|_{1+i}^{-i}$$

can use FTOC since conditions are satisfied

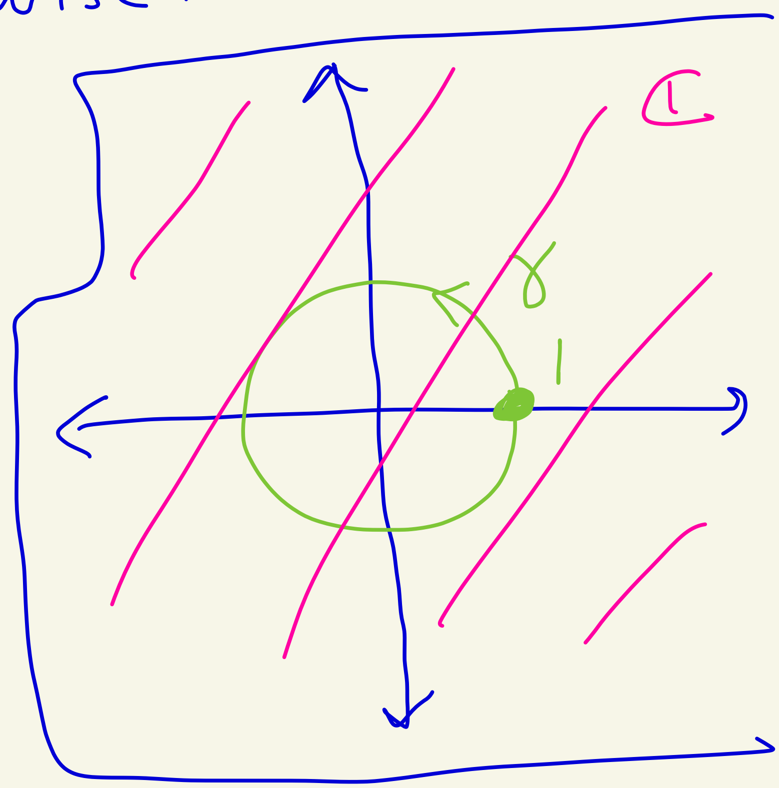
$$= \boxed{\tfrac{1}{2}\sin(-2i) - \tfrac{1}{2}\sin(2+2i)}$$

$\boxed{2}$ (b)

Let γ be the unit circle traveled once counterclockwise.

Let $F(z) = \frac{1}{2}e^{z^2}$

and $F'(z) = ze^{z^2}$.



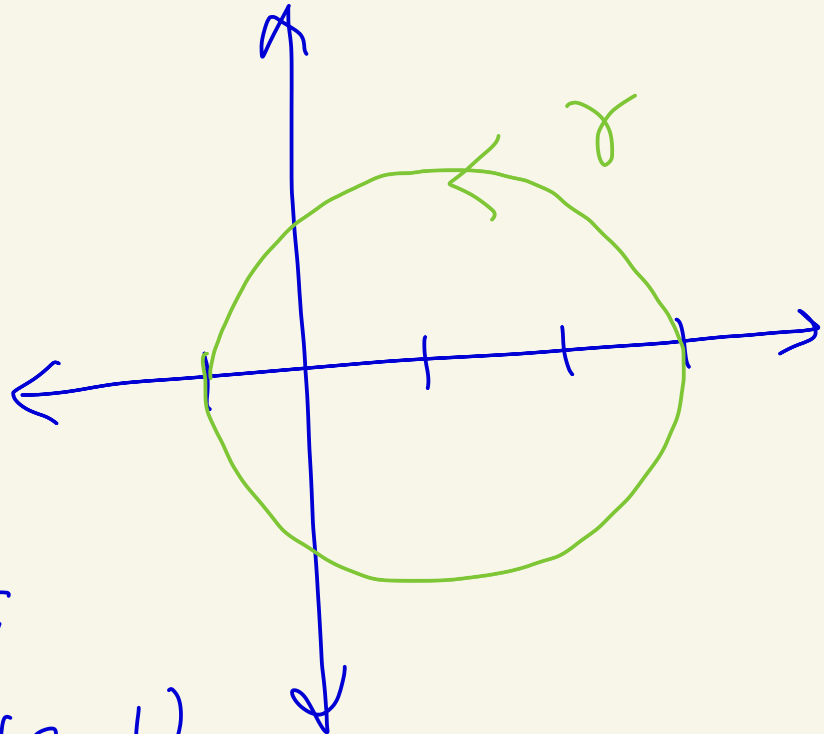
Note that $F(z)$ is analytic on the open set $\mathbb{C}$ containing γ and $F'(z)$ is continuous on $\mathbb{C}$.

Thus, by the FTOC,

$$\int_\gamma ze^{z^2}\,dz = \int_\gamma F'(z)\,dz$$

$$= \underbrace{F(1) - F(1)}_{\gamma \text{ starts and ends at the same point}} = 0$$

$\boxed{2}$ (c)



Note that we cannot use the FTOC here because

If $F'(z) = \frac{1}{z-1}$

then $F(z) = \log(z-1)$ which is not analytic on any open set cointaing γ since γ world cut through the branch cut.

So we have to do this one by hand.

$\gamma(t) = 1 + 2e^{it}, \quad 0 \le t \le 2\pi$

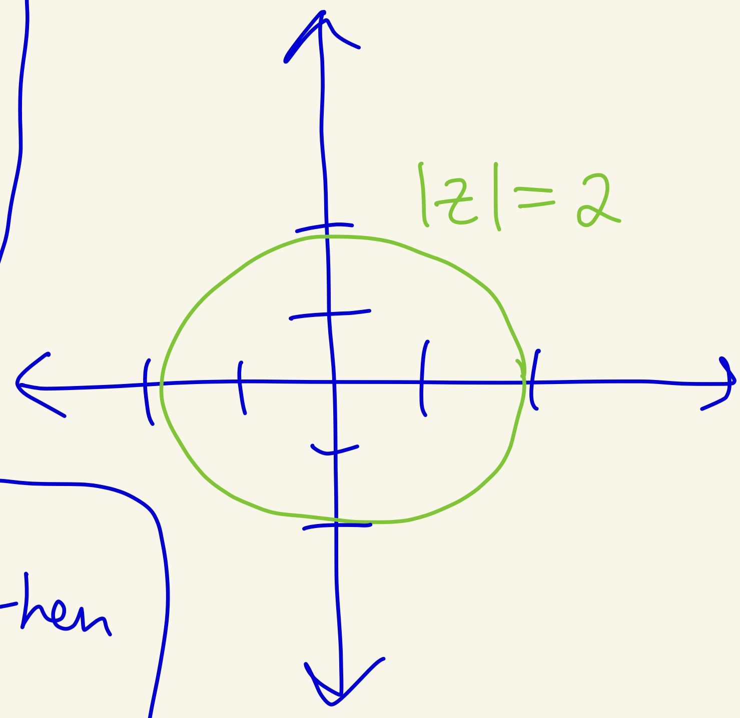
$\gamma'(t) = 2ie^{t}$

$$\int_\gamma \frac{dz}{z-1} = \int_0^{2\pi} \frac{2ie^{it}}{(1+2e^{it})-1}\,dt = \int_0^{2\pi} i\,dt$$

$$= i\int_0^{2\pi} 1\,dt = i\,z\Big|_0^{2\pi} = 2\pi i$$

Think this: To show $\frac{a}{b} \le \frac{c}{d}$ you show $a \le c$ and $b \ge d$. So then $\frac{1}{b} \le \frac{1}{d}$ and so $\frac{a}{b} \le \frac{c}{d}$.

(3) Note that if z is on γ then $|z| = 2$.



So if z is on γ, then

$$|z^2+1| \ge \big||z^2|-|1|\big| = \big||z|^2-1\big| = |2^2-1| = 3$$

$|a+b| \ge \big||a|-|b|\big|$

Let $M = \frac{1}{3}$. If z is on γ then

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|} \le \frac{1}{3} = M.$$

$|z^2+1| \ge 3$

$\frac{1}{|z^2+1|} \le \frac{1}{3}$

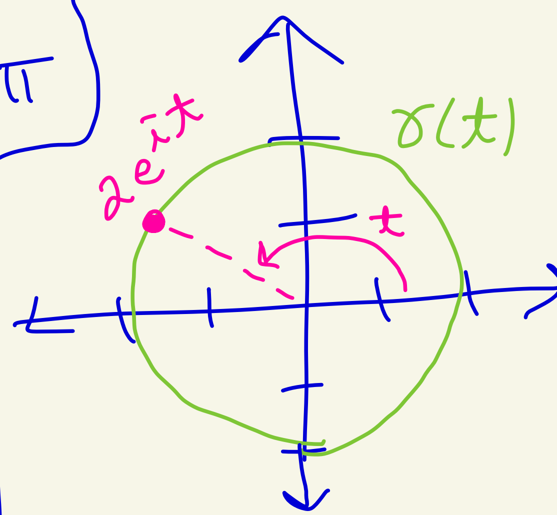
Thus,

$$\left|\int_\gamma \frac{dz}{z^2+1}\right| \le M \cdot \text{arclength}(\gamma) = \frac{1}{3}\,\text{arclength}(\gamma)$$

Here the arclength is the circumference of the circle. So, $\text{arclength}(\gamma) = 2\pi(2)$
$= 4\pi$.

Let's use the def of arclength to show this.

$$\gamma(t) = 2e^{it}, \quad 0 \le t \le 2\pi$$



And so

$$\text{arclength}(\gamma) = \int_0^{2\pi} |\gamma'(t)|\,dt$$

$$= \int_0^{2\pi} |2ie^{it}|\,dt = \int_0^{2\pi} 2\,dt = 4\pi$$

Ok, thus,

$$\left| \int_\gamma \frac{dz}{z^2+1} \right| \le \frac{1}{3}\,\text{arclength}(\gamma) = \frac{4\pi}{3}$$

$\blacksquare$

Think this: To show $\frac{a}{b} \le \frac{c}{d}$ you show $a \le c$ and $b \ge d$. So then $\frac{1}{b} \le \frac{1}{d}$ and so $\frac{a}{b} \le \frac{c}{d}$.

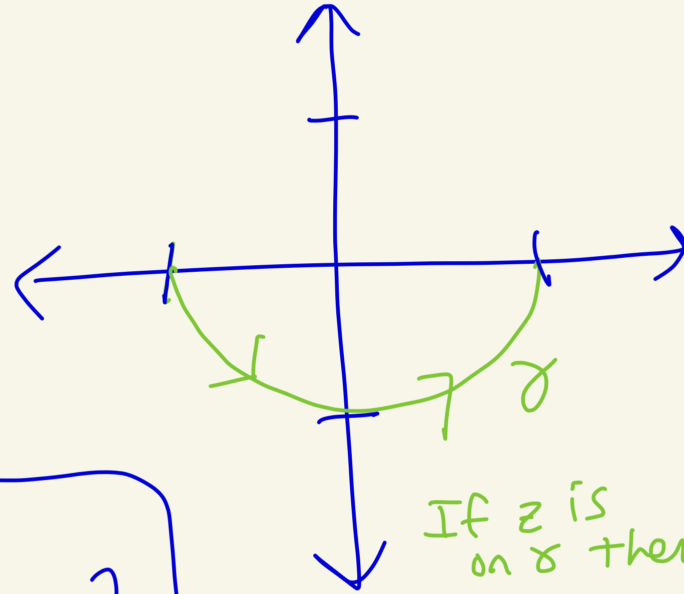
4

Suppose z is on γ.

Then, $|z| = 1$

and thus

$$|z+1| \le |z| + |1| = 1 + 1 = 2$$



If z is on γ then $|z|=1$

and

$$|z^2 - 8| \ge \big|\, |z^2| - |8| \,\big| = \big|\, |z|^2 - 8 \,\big|$$

$$= |1^2 - 8|$$

$$= |-7| = 7$$

$|a+b| \ge \big|\, |a| - |b| \,\big|$

So, $\dfrac{1}{|z^2-8|} \le \dfrac{1}{7}$.

Thus, if z is on γ, then

$$\left|\frac{z+1}{z^2-8}\right| = \frac{|z+1|}{|z^2-8|} \le \frac{2}{7}.$$

because $|z+1| \le 2$

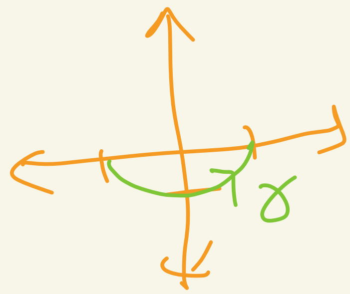
$\dfrac{1}{|z^2-8|} \le 7$

Let $M = \frac{2}{7}$.

Then,

$$\left|\int_\gamma \frac{z+1}{z^2-8}\,dz\right| \le M \operatorname{arclength}(\gamma).$$
$$= \frac{2}{7}\operatorname{arclength}(\gamma)$$

You can calculate the arclength as the length of γ with is $\frac{1}{2}2\pi(1) = \pi$ or by the integral:



Let $\gamma(t) = e^{it}$, $\pi \le t \le 2\pi$

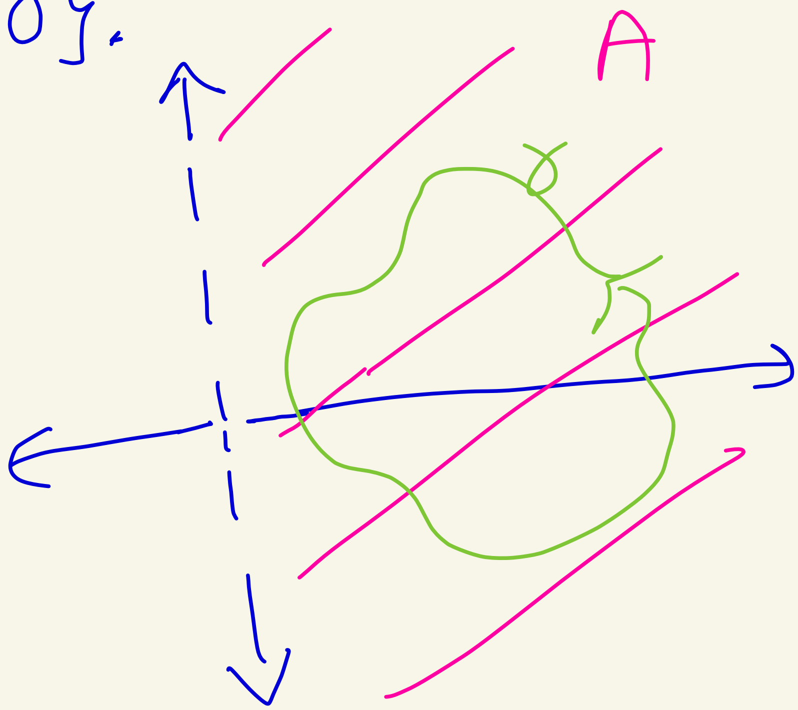
$$\operatorname{arclength}(\gamma) = \int_\pi^{2\pi} |\gamma'(t)|\,dt$$
$$= \int_\pi^{2\pi} |ie^{it}|\,dt = \int_\pi^{2\pi} 1\,dt$$
$$= \pi$$

Thus

$$\left|\int_\gamma \frac{z+1}{z^2-8}\,dz\right| \le \frac{2}{7}\pi \qquad \blacksquare$$

5 (a)

Let γ be a closed piecewise smooth curve lying entirely in $A = \mathbb{C} - \{z \mid \operatorname{Re}(z) \leq 0\}$.



Let $\log(z)$ be the principal branch of the logarithm.

Then $\log(z)$ is analytic on A,

And $\frac{1}{z}$ is continuous on A since $0 \notin A$.

Since γ is closed, $\gamma(a) = \gamma(b)$

where $\gamma : [a, b] \to \mathbb{C}$,

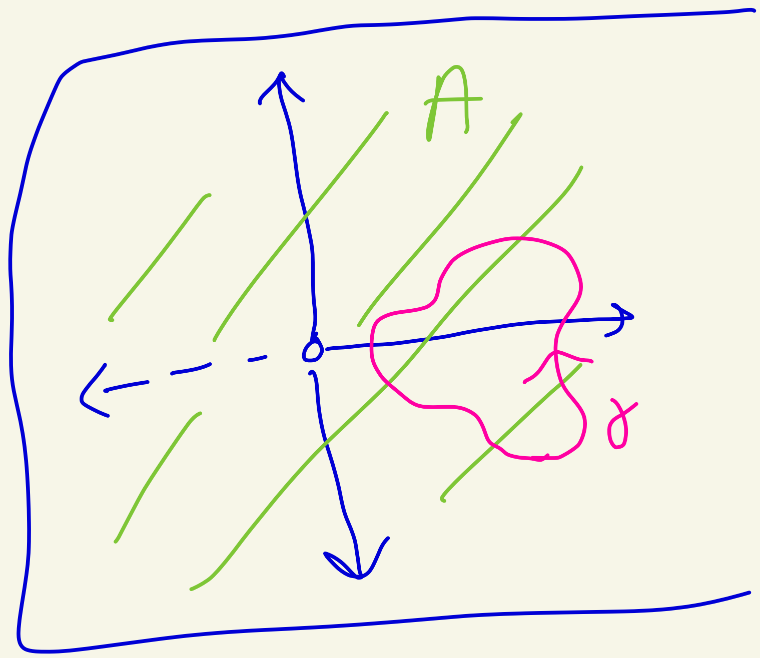
So, by the FTOC,

$$\int_\gamma \underbrace{\frac{dz}{z}}_{\substack{1/z \\ \text{continuous on } A}} = \underbrace{\log(\gamma(b)) - \log(\gamma(a))}_{\log(z)\text{ analytic on } A}$$

FTOC conditions met

$$= \log(\gamma(a)) - \log(\gamma(a)) = 0$$

(5) (b) Let $A = \mathbb{C} - \{x+iy \mid x \leq 0, y=0\}$ and $\gamma : [a,b] \to \mathbb{C}$. Here $\gamma(b) = \gamma(a)$.



If γ is totally contained in A since $F(z) = \log(z)$ is analytic on A (where $\log(z)$ is the principal branch of the logarithm) and $F'(z) = \frac{1}{z}$ is continuous on A, by the FTOC we get that

$$\int_\gamma \frac{dz}{z} = \log(\gamma(b)) - \log(\gamma(a))$$

$$= \log(\gamma(a)) - \log(\gamma(a))$$

$$= 0.$$