Math 5800
HW 7 Solutions

The Lebesgue Integral

1 (a)

Let f be a step function.

Define the sequence $(\varphi_n)_{n=1}^{\infty}$ of step functions to be the constant sequence $\varphi_n = f$ for all $n \geq 1$,

ie the sequence

$$f, f, f, f, f, \ldots$$
$$\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \ldots$$

This is a non-decreasing sequence since $\varphi_n = \varphi_{n+1}$ for all $n \geq 1$.

Also, $\lim\limits_{n\to\infty} \varphi_n(x) = \lim\limits_{n\to\infty} f(x) = f(x)$ for all $x \in \mathbb{R}$.

Since $\varphi_n = f$ is a step function we know that $\int \varphi_n = \int f$ is some finite real number.

Thus, $\lim_{n\to\infty} \int \varphi_n = \int f$ converges.

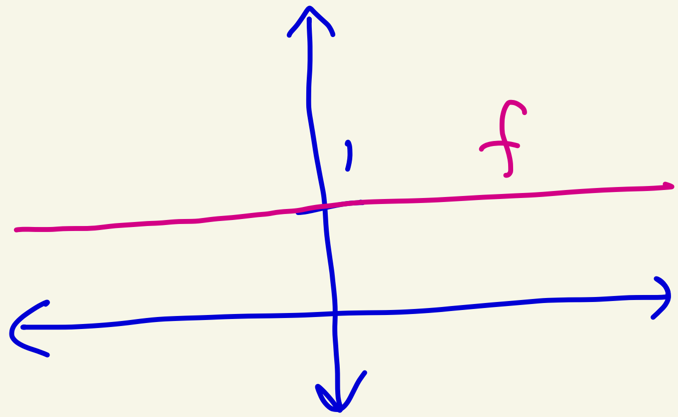
We have satisfied all the properties that are needed to show that $f \in L^0$.

① (b) Let f be a step function.
Then $f \in L^0$ by 1(a).
Since $L^0 \subseteq L^1$ we have
that $f \in L^1$.

$\boxed{//}$

② (a)

Let $f = \chi_{\mathbb{R}}$



We will show that

$f \notin L^1$.

We do this by contradiction.

Suppose $f \in L^1$.

Then $\int f$ is some real number.

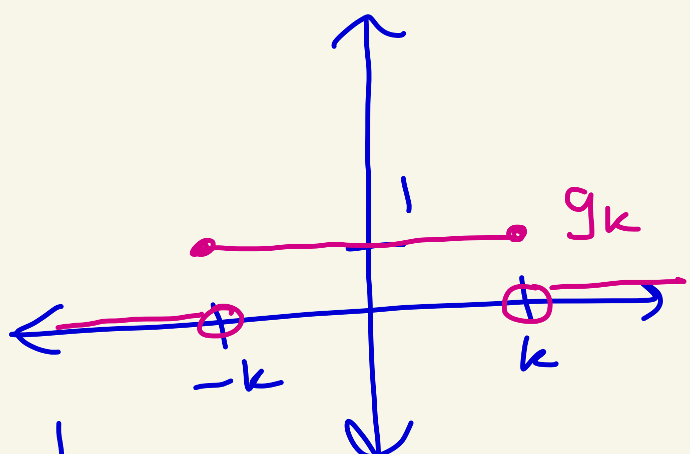
Define the step function

$$g_k(x) = \begin{cases} 1 & \text{if } x \in [-k,k] \\ 0 & \text{otherwise} \end{cases}$$

Since $g_k = \chi_{[-k,k]}$

we have that

g_k is a step function.



From problem 1, $g_k \in L^1$

for each $k \geq 1$.

Let's show that $g_k(x) \leq f(x)$
for all $x \in \mathbb{R}$.

If $x \in [-k, k]$, then $g_k(x) = 1 = f(x)$.
If $x \notin [-k, k]$, then $g_k(x) = 0 < 1 = f(x)$.
Thus, $g_k(x) \leq f(x)$ for all $x \in \mathbb{R}$.
Therefore, from class $\int g_k \leq \int f$
for all $k \geq 1$.

Also, $\int g_k = \int \chi_{[-k,k]} = 1 \cdot [k - (-k)]$
$= 2k$.

Thus, $2k \leq \int f$ for all $k \geq 1$.
But $2k \to \infty$ as $k \to \infty$, thus
$\int f$ could not be a real number.

Contradiction.
Thus, $f \notin L^1$.

(2) (b) Let I be a finite interval.

Let $g = \chi_I \cdot f$

Since $f(x) = 1$ for all $x \in \mathbb{R}$ this gives that $g = \chi_I$.

Since χ_I is a step function, we have that $g \in L^1$.

Thus, $f \in L^1(I)$ by the def of $L^1(I)$.

$\boxtimes$

③ (a) and (b) together

Let $f, g \in L^0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$.

Since $f \in L^0$ there exists a non-decreasing sequence of step functions $(f_n)_{n=1}^{\infty}$ that converge to f on an almost everywhere set A.
And $\lim_{n\to\infty} \int f_n$ converges.
So that $\int f = \lim_{n\to\infty} \int f_n$.

Since $g \in L^0$ there exists a non-decreasing sequence of step functions $(g_n)_{n=1}^{\infty}$ that converge to g on an almost everywhere set B.
And $\lim_{n\to\infty} \int g_n$ converges.
So that $\int g = \lim_{n\to\infty} \int g_n$.

We know that $f_n(x) \leq f_{n+1}(x)$ and $g_n(x) \leq g_{n+1}(x)$ for all $n \geq 1$ and $x \in \mathbb{R}$.

Since $\alpha, \beta \geq 0$ we have that $\alpha f_n(x) \leq \alpha f_{n+1}(x)$ and $\beta g_n(x) \leq \beta g_{n+1}(x)$ for all $n \geq 1$ and $x \in \mathbb{R}$.

Thus,

$$\alpha f_n(x) + \beta g_n(x) \leq \alpha f_{n+1}(x) + \beta g_{n+1}(x)$$

for all $x \in \mathbb{R}$ and $n \geq 1$.

So, $(\alpha f_n + \beta g_n)$ is a non-decreasing sequence of step functions.

Since $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$ we have that $\lim_{n\to\infty} \alpha f_n(x) = \alpha f(x)$

for all $x \in A$.

Similarly, $\lim_{n\to\infty} \beta g_n(x) = \beta g(x)$ for all $x \in B$

Thus,

$$\lim \left(\alpha f_n(x) + \beta g_n(x)\right) = \alpha f(x) + \beta g(x)$$

for all $x \in A \cap B$.

Since A and B are almost everywhere sets, from HW, $A \cap B$ is an almost everywhere set.

Thus, $\alpha f_n + \beta g_n \to \alpha f + \beta g$ almost everywhere.

By the property of step functions

$$\lim_{n\to\infty} \int \alpha f_n + \beta g_n = \alpha \lim_{n\to\infty} \int f_n + \beta \lim_{n\to\infty} \int g_n$$

$$= \alpha \int f + \beta \int g$$

So, $(\alpha f_n + \beta g_n)_{n=1}^{\infty}$ is a sequence of non-decreasing step functions with $\alpha f_n + \beta g_n \to \alpha f + \beta g$ almost everywhere,

and $\lim_{n\to\infty} \int \alpha f_n + \beta g_n$ converges.

Thus, $\alpha f + \beta g \in L^0$ and

$$\int \alpha f + \beta g = \lim_{n\to\infty} \int \alpha f_n + \beta g_n$$

$$= \alpha \int f + \beta \int g$$

$\boxed{/\!/}$

4 Let $f \in L^0$ and $g: \mathbb{R} \to \mathbb{R}$. Further suppose that $f(x) = g(x)$ for all $x \in A$ where A is an almost everywhere set.

Since $f \in L^0$ there existr a non-decreasing sequence of step functions $(\varphi_n)_{n=1}^{\infty}$ where $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in B$ where B is an almost everywhere set, and $\lim_{n\to\infty} \int \varphi_n$ converges, and $\int f = \lim_{n\to\infty} \int \varphi_n$.

Since A and B are almost everywhere sets, $A \cap B$ is an almost everywhere set.

And if $x \in A \cap B$ then

$$\lim_{n\to\infty} \varphi_n(x) = \underbrace{}_{x\in B}\!\! f(x) = \underbrace{}_{x\in A}\!\! g(x)$$

Thus, $(\varphi_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions with $\varphi_n \to g$ almost everywhere.

And $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges.

Thus, $g \in L^0$ and $\int g = \lim_{n\to\infty} \int \varphi_n = \int f$.

$\blacksquare$

5

(a) Since $\int \varphi_n \leq M$ for all n, from a theorem in class, because $(\varphi_n)_{n=1}^{\infty}$ is a non-decreasing sequence we know that $\lim_{n\to\infty} \int \varphi_n$ converges.

Since $\varphi_n \to f$ almost everywhere, $f \in L^0$.

5 (b) From (a) and the def of L^0, we have that

$$\int f = \lim_{n\to\infty} \int \varphi_n$$

5 (c)

We know that $\int \varphi_n \leq M$ for $n \geq 1$.

Thus, from 4650 HW2,

$$\lim_{n \to \infty} \int \varphi_n \leq \lim_{n \to \infty} M$$

Thus, $\int f \leq M$.

$\blacksquare$

⑥ We need a claim:

Claim: $f \cdot \chi_{[a,b]} = f \cdot \chi_{[a,c]} + f \cdot \chi_{[c,b]}$ almost everywhere

pf of claim:

If $x \notin [a,b]$, then

$$\left(f \cdot \chi_{[a,b]}\right)(x) = f(x) \cdot \chi_{[a,b]}(x) = f(x) \cdot 0 = 0$$

and

$$\left(f \cdot \chi_{[a,c]}\right)(x) + \left(f \cdot \chi_{[c,b]}\right)(x)$$
$$= f(x)\,\chi_{[a,c]}(x) + f(x)\,\chi_{[c,b]}(x)$$
$$= f(x) \cdot 0 + f(x) \cdot 0 = 0$$

Thus, the claim is true for $x \notin [a,b]$.

Now suppose $x \in [a,b]$, but $x \neq c$.

<u>case 1: Suppose $a \leq x < c$</u>

Then,

$$\left(f \cdot \chi_{[a,b]}\right)(x) = f(x) \underbrace{\chi_{[a,b]}(x)}_{1} = f(x)$$

and

$$\left(f \cdot \chi_{[a,c]}\right)(x) + \left(f \cdot \chi_{[c,b]}\right)(x)$$

$$= f(x) \underbrace{\chi_{[a,c]}(x)}_{1} + f(x) \underbrace{\chi_{[c,b]}}_{0}(x)$$

$$= f(x)$$

<u>case 2: Suppose $c < x \leq b$</u>

Then,

$$\left(f \cdot \chi_{[a,b]}\right)(x) = f(x) \underbrace{\chi_{[a,b]}(x)}_{1} = f(x)$$

and

$$\left(f\cdot\chi_{[a,c]}\right)(x)+\left(f\cdot\chi_{[c,b]}\right)(x)$$

$$=f(x)\underbrace{\chi_{[a,c]}(x)}_{0}+f(x)\underbrace{\chi_{[c,b]}(x)}_{1}$$

$$=f(x)$$

Thus, from the above $f\cdot\chi_{[a,b]}=f\cdot\chi_{[a,c]}+f\cdot\chi_{[c,b]}$ on $\mathbb{R}-\{c\}$, so almost everywhere. $\boxed{\text{Claim}}$

Using the claim we have that

$$\int_a^b f=\int f\cdot\chi_{[a,b]}$$

By claim

$$=\int f\cdot\chi_{[a,c]}+f\cdot\chi_{[c,b]}$$

$$=\int f\cdot\chi_{[a,c]}+\int f\cdot\chi_{[c,b]}$$

$$=\int_a^c f+\int_c^b f$$

$\blacksquare$

7

Let $x \in \mathbb{R}$.

If $x \notin [a,b]$, then

$$m \cdot \chi_{[a,b]} = m \cdot 0 = 0$$

$$f(x) \cdot \chi_{[a,b]}(x) = f(x) \cdot 0 = 0$$

and $M \cdot \chi_{[a,b]} = M \cdot 0 = 0$.

Thus,

$$m \cdot \chi_{[a,b]}(x) \leq f(x) \cdot \chi_{[a,b]}(x) \leq M \cdot \chi_{[a,b]}(x)$$

If $x \in [a,b]$, then $m \leq f(x) \leq M$ and so by multiplying by $\chi_{[a,b]}(x)$ we get that

$$m \cdot \chi_{[a,b]}(x) \leq f(x) \cdot \chi_{[a,b]}(x) \leq M \cdot \chi_{[a,b]}(x)$$

Thus, $m \cdot \chi_{[a,b]} \leq f \cdot \chi_{[a,b]} \leq M \cdot \chi_{[a,b]}$.

Since f is integrable on $[a,b]$ we have that $f \cdot \chi_{[a,b]} \in L^1$.

Since $m \cdot \chi_{[a,b]}$ and $M \cdot \chi_{[a,b]}$ are step functions we have that $m \cdot \chi_{[a,b]} \in L^1$ and $M \cdot \chi_{[a,b]} \in L^1$.

Thus, since

$$m \cdot \chi_{[a,b]} \leq f \cdot \chi_{[a,b]} \leq M \cdot \chi_{[a,b]}$$

we know that

$$\int m \cdot \chi_{[a,b]} \leq \int f \cdot \chi_{[a,b]} \leq \int M \cdot \chi_{[a,b]}$$

Thus,

$$m \cdot (b-a) \leq \int_a^b f \leq M \cdot (b-a)$$

$\blacksquare$

8 (a)

Break the interval $[-1,1]$ into 2^n subintervals of width $\Delta_n = \frac{1-(-1)}{2^n} = \frac{1}{2^{n-1}}$.

These are

$$I_{n,1} = \left[-1+0\cdot\frac{1}{2^{n-1}}, -1+1\cdot\frac{1}{2^{n-1}}\right)$$

$$I_{n,2} = \left[-1+1\cdot\frac{1}{2^{n-1}}, -1+2\cdot\frac{1}{2^{n-1}}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_{n,k} = \left[-1+(k-1)\frac{1}{2^{n-1}}, -1+k\cdot\frac{1}{2^{n-1}}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_{n,2^n} = \left[-1+(2^n-1)\cdot\frac{1}{2^{n-1}}, -1+2^n\cdot\frac{1}{2^{n-1}}\right]$$

Since $x+1$ is an increasing function on $[-1,1]$, the infimum of $x+1$ on $I_{n,k}$ is the left endpoint.

Thus,

$$\gamma_n = f(-1)\cdot \chi_{I_{n,1}} + f\left(-1+\frac{1}{2^{n-1}}\right)\cdot \chi_{I_{n,2}}$$

$$+ f\left(-1+2\cdot\frac{1}{2^{n-1}}\right)\cdot \chi_{I_{n,3}} + f\left(-1+3\cdot\frac{1}{2^{n-1}}\right)\cdot \chi_{I_{n,4}}$$

$$+ \dots + f\left(-1+(k-1)\cdot\frac{1}{2^{n-1}}\right)\cdot \chi_{I_{n,k}}$$

$$+ \dots + f\left(-1+\frac{2^n-1}{2^{n-1}}\right)\cdot \chi_{I_{n,2^n}}$$

$$= 0\cdot \chi_{I_{n,1}} + \frac{1}{2^{n-1}}\cdot \chi_{I_{n,2}} + 2\cdot\frac{1}{2^{n-1}}\cdot \chi_{I_{n,3}}$$

$$\boxed{f(x)=x+1} \quad + 3\cdot\frac{1}{2^{n-1}}\cdot \chi_{I_{n,4}} + \dots + (2^n-1)\cdot\frac{1}{2^{n-1}}\cdot \chi_{I_{n,2^n}}$$

We have that

$$\ell(I_{n,k}) = \Delta_n = \frac{1}{2^{n-1}}$$

So,

$$\int \gamma_n = (0)\cdot \frac{1}{2^{n-1}} + \left(\frac{1}{2^{n-1}}\right)\cdot \frac{1}{2^{n-1}} + \left(2\cdot \frac{1}{2^{n-1}}\right)\cdot \frac{1}{2^{n-1}} + \cdots + \left(\frac{2^n-1}{2^{n-1}}\right)\cdot \frac{1}{2^{n-1}}$$

$$= \sum_{k=1}^{2^n} \left((k-1)\cdot \frac{1}{2^{n-1}}\right)\cdot \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}}\cdot \frac{1}{2^{n-1}}\cdot \sum_{k=1}^{2^n} (k-1)$$

$$= \left(\frac{1}{2^{n-1}}\right)^2 \cdot \sum_{l=1}^{2^n-1} l$$

$$\boxed{1+2+3+\dots+m = \frac{m(m+1)}{2}}$$

$$= \left(\frac{1}{2^{n-1}}\right)^2 \cdot \left[\frac{(2^n-1)(2^n-1+1)}{2}\right]$$

$$= \left(\frac{1}{2^{n-1}}\right)^2 \cdot \left[(2^n-1)(2^{n-1})\right] = \frac{2^n-1}{2^{n-1}}$$

8 (b) We have that

γ_n is a non-decreasing sequence of step functions with $\gamma_n \to f$ pointwise on all of $\mathbb{R}$.

Also,

$$\lim_{n\to\infty} \int \gamma_n = \lim_{n\to\infty} \left[\frac{2^n-1}{2^{n-1}}\right] = \lim_{n\to\infty} \left[2 - \frac{1}{2^{n-1}}\right]$$

$$= 2$$

Thus, $f \in L^1$ and

$$\int f = \lim_{n\to\infty} \int \gamma_n = 2$$

8 (c)

Let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x+1$.

Then, $g \cdot \chi_{[-1,1]} = f \in L^1$.

Thus, $g \in L^1([-1,1])$ and

$$\int_{-1}^{1} (x+1)\,dx = \int_{-1}^{1} g = \int g \cdot \chi_{[-1,1]}$$

$$= \int f = 2$$

9 (a)

Break the interval $[0,1]$ into 2^n subintervals of width $\Delta_n = \frac{1-0}{2^n} = \frac{1}{2^n}$

These are

$$I_{n,1} = \left[0 + 0\cdot\frac{1}{2^n},\ 0 + 1\cdot\frac{1}{2^n}\right) = \left[0, \frac{1}{2^n}\right)$$

$$I_{n,2} = \left[0 + 1\cdot\frac{1}{2^n},\ 0 + 2\cdot\frac{1}{2^n}\right) = \left[\frac{1}{2^n}, \frac{2}{2^n}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_{n,k} = \left[0 + (k-1)\frac{1}{2^n},\ 0 + k\cdot\frac{1}{2^n}\right) = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_{n,2^n} = \left[0 + (2^n-1)\cdot\frac{1}{2^n},\ 0 + 2^n\cdot\frac{1}{2^n}\right] = \left[\frac{2^n-1}{2^n}, \frac{2^n}{2^n}\right]$$

Since x^2 is an increasing function on $[0,1]$, the infimum of x^2 on $I_{n,k}$ is the left endpoint.

Thus,

$$\gamma_n = f(0)\cdot \chi_{I_{n,1}} + f\left(\frac{1}{2^n}\right)\cdot \chi_{I_{n,2}}$$

$$+ f\left(2\cdot\frac{1}{2^n}\right)\cdot \chi_{I_{n,3}} + f\left(3\cdot\frac{1}{2^n}\right)\cdot \chi_{I_{n,4}}$$

$$+ \dots + f\left((k-1)\cdot\frac{1}{2^n}\right)\cdot \chi_{I_{n,k}}$$

$$+ \dots + f\left(\frac{2^n-1}{2^n}\right)\cdot \chi_{I_{n,2^n}}$$

$$= 0^2\cdot \chi_{I_{n,1}} + \left(\frac{1}{2^n}\right)^2\cdot \chi_{I_{n,2}} + \left(\frac{2}{2^n}\right)^2\cdot \chi_{I_{n,3}}$$

$\boxed{f(x) = x^2}$

$$+ \left(\frac{3}{2^n}\right)^2\cdot \chi_{I_{n,4}} + \dots + \left(\frac{2^n-1}{2^n}\right)^2\cdot \chi_{I_{n,2^n}}$$

We have that

$$\ell(I_{n,k}) = \Delta_n = \frac{1}{2^n}$$

Thus,

$$\int \gamma_n = 0^2 \cdot \frac{1}{2^n} + \left(\frac{1}{2^n}\right)^2 \cdot \frac{1}{2^n} + \left(\frac{2}{2^n}\right)^2 \cdot \frac{1}{2^n} + \dots + \left(\frac{2^n-1}{2^n}\right)^2 \cdot \frac{1}{2^n}$$

$$= \frac{1}{2^n} \cdot \sum_{k=0}^{2^n-1} \left(\frac{k}{2^n}\right)^2$$

$$= \left(\frac{1}{2^n}\right)^3 \cdot \sum_{l=1}^{2^n-1} l^2$$

$$\boxed{1+2+3+\dots+m = \frac{m(m+1)}{2}}$$

$$= \left(\frac{1}{2^n}\right)^3 \cdot \left[\frac{(2^n-1)\cdot(2^n-1+1)(2\cdot(2^n-1)+1)}{6}\right]$$

$$= \frac{1}{(2^n)^3} \cdot \frac{1}{6} \cdot (2^n-1)(2^n)(2^{n+1}-1)$$

$$= \frac{(2^n-1)(2^{n+1}-1)}{6\cdot(2^n)^2} =$$

$$= \frac{2^{2n+1} - 2^n - 2^{n+1} + 1}{6 \cdot 2^{2n}}$$

$$= \frac{2 \cdot 2^{2n} - 3 \cdot 2^n + 1}{6 \cdot 2^{2n}}$$

$$\boxed{2^{n+1} = 2 \cdot 2^n}$$

9 (b) We know that γ_n is a non-decreasing sequence that converges pointwise to f on $\mathbb{R}$.

Also,

$$\lim_{n \to \infty} \int \gamma_n = \lim_{n \to \infty} \frac{2 \cdot 2^{2n} - 3 \cdot 2^n + 1}{6 \cdot 2^{2n}} =$$

$$= \lim_{n\to\infty} \frac{2 - 3\cdot\frac{1}{2^n} + \frac{1}{2^{2n}}}{6}$$

divide top/bottom by 2^{2n}

$$= \frac{2-3\cdot 0 + 0}{6} = \frac{1}{3}$$

9 (c)

Let $g:\mathbb{R}\to\mathbb{R}$ be $g(x)=x^2$

Then, $g\cdot \chi_{[0,1]} = f \in L^1$.

Thus, $g \in L^1([0,1])$ and

$$\int_0^1 x^2\,dx = \int_0^1 g = \int g\cdot\chi_{[0,1]}$$

$$= \int f = \frac{1}{3}$$

(10) Let I be a bounded interval.

Let $h = g \cdot \chi_I$.

Then,

$$h(x) = \begin{cases} 1 & \text{if } x \text{ is irrational and } x \in I \\ 0 & \text{otherwise} \end{cases}$$

We want to show that $h \in L^1$ and this will show that $g \in L^1(I)$.

Consider the constant sequence

$$\varphi_n = \chi_I$$

for all $n \geq 1$, ie the sequence

$$\chi_I, \chi_I, \chi_I, \chi_I, \ldots$$

Then, if $x \in I$ and x is irrational) we have that

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \chi_I(x) = \lim_{n\to\infty} 1$$

$$= 1 = h(x)$$

If $x \in I$ and x is rational then

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \chi_I(x) = \lim_{n\to\infty} 1$$

$$= 1 \neq 0 = h(x).$$

If $x \notin I$, then

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \chi_I(x) = \lim_{n\to\infty} 0$$

$$= 0 = h(x)$$

So, $\varphi_n \to h$ everywhere except on $\mathbb{Q} \cap I$ which has measure zero.

So, $\varphi_n \to h$ almost everywhere.

Since $(\varphi_n)_{n=1}^{\infty}$ is a constant sequence it is non-decreasing.

And, $\lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \ell(I) = \ell(I)$

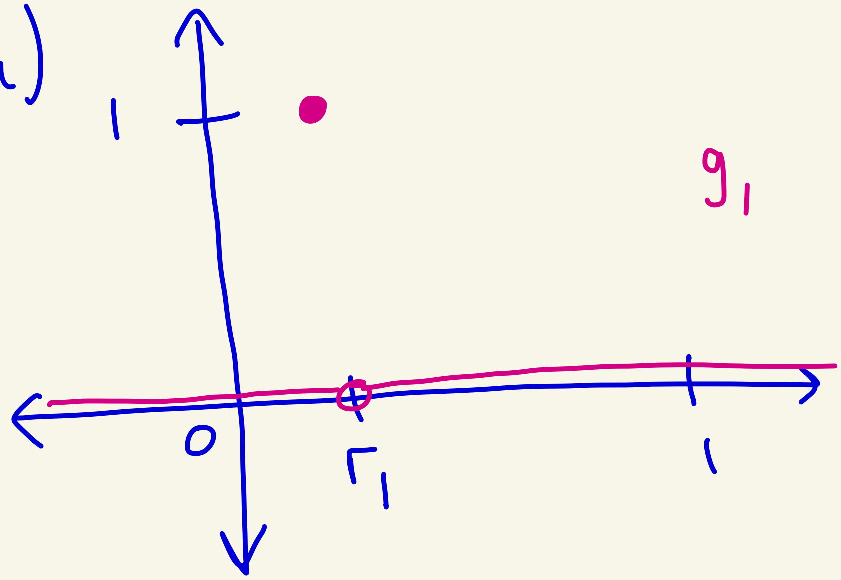
So, $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges.

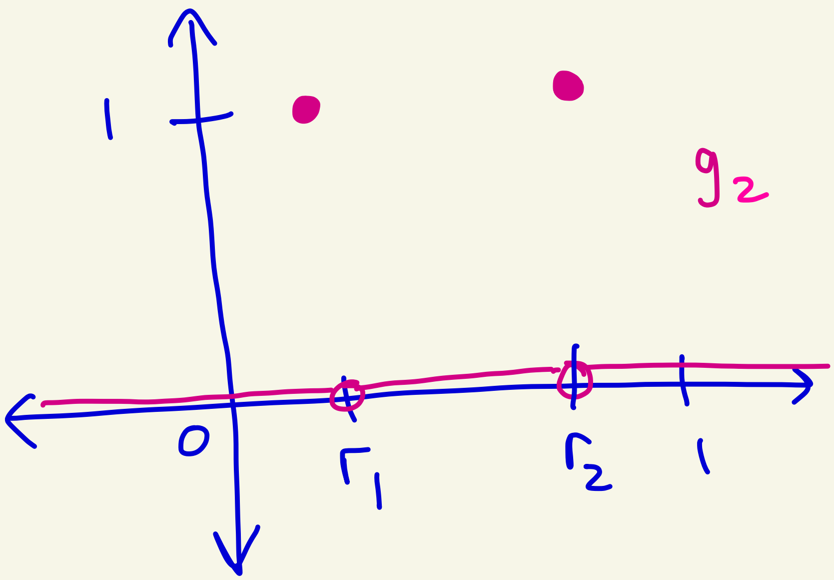
Thus, $h \in L^1$ and $\int h = \lim_{n\to\infty} \int \varphi_n = \ell(I)$.

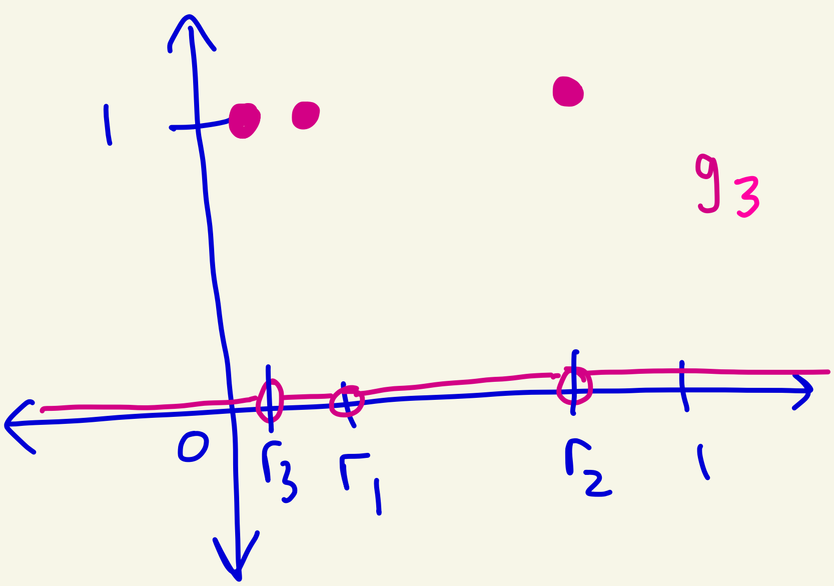
So, $g \in L^1(I)$ and

$$\int_I g = \int g \cdot \chi_I = \int h = \ell(I).$$

(11) (a)







11 (b) Let $n \geq 1$ be fixed.

Then,

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise} \end{cases}$$

So,

$$g_n = \chi_{[r_1,r_1]} + \chi_{[r_2,r_2]} + \ldots + \chi_{[r_n,r_n]}$$

Thus, g_n is a step function.

We have that

$$g_{n+1}(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n, r_{n+1}\} \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$g_{n+1} = \chi_{[r_1, r_1]} + \chi_{[r_2, r_2]} + \dots$$
$$\dots + \chi_{[r_n, r_n]} + \chi_{[r_{n+1}, r_{n+1}]}$$

$$= g_n + \chi_{[r_{n+1}, r_{n+1}]}$$

Therefore, for any $x \in \mathbb{R}$ we have that

$$g_{n+1}(x) = g_n(x) + \chi_{[r_{n+1}, r_{n+1}]}(x)$$

$$\geq g_n(x)$$

Since $\chi_{[r_{n+1}, r_{n+1}]}(x) \geq 0$

So, $(g_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions.

11 (c)

We show that $g_n \to g$ on $\mathbb{R}$.

Let $x \in \mathbb{R}$.

Case 1: Suppose $x \notin [0,1]$.

Then, $g_n(x) = 0 = g(x)$ for all n.

Thus,
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} 0 = 0 = g(x).$$

Case 2: Suppose $x \in [0,1]$ and x is irrational

Then, $g_n(x) = 0 = g(x)$ for all n.

Thus,
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} 0 = 0 = g(x)$$

Case 3: Suppose $x = r_k$ for some $k \geq 1$.

Choose $N = k$.
Then if $n \geq N = k$ we have that

$$g_n(x) = g_n(r_k) = 1.$$

Let $\varepsilon > 0$.
Then if $n \geq N$ we have that

$$|g_n(x) - g(x)| = |g_n(r_k) - g(r_k)|$$
$$= |1 - 1| = 0 < \varepsilon.$$

Thus, $g_n(x) \longrightarrow g(x)$.

By case 1, case 2, case 3 we have that $g_n \to g$ converges pointwise on all of $\mathbb{R}$.

(11) (d)

$$\int g_n = \int \chi_{[r_1,r_1]} + \chi_{[r_2,r_2]} + \dots + \chi_{[r_n,r_n]}$$

$$= 1\cdot(r_1 - r_1) + 1\cdot(r_2 - r_2) + \dots + 1\cdot(r_n - r_n)$$

$$= 1\cdot 0 + 1\cdot 0 + \dots + 1\cdot 0 = 0$$

(11) (e)

Since $(g_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions that converges pointwise on all of $\mathbb{R}$ to g and $\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} 0 = 0$ converges we have that $g \in L^0$ and

$$\int g = \lim_{n\to\infty} \int g_n = 0.$$

(12) We induct on s.

Suppose $s=1$.

Then, $T_1 \subseteq [a,b]$.

Since T_1 is a bounded interval, T_1 is of the form $[c,d]$, $[c,d)$, $(c,d]$, or (c,d) where $a \le c \le d \le b$.

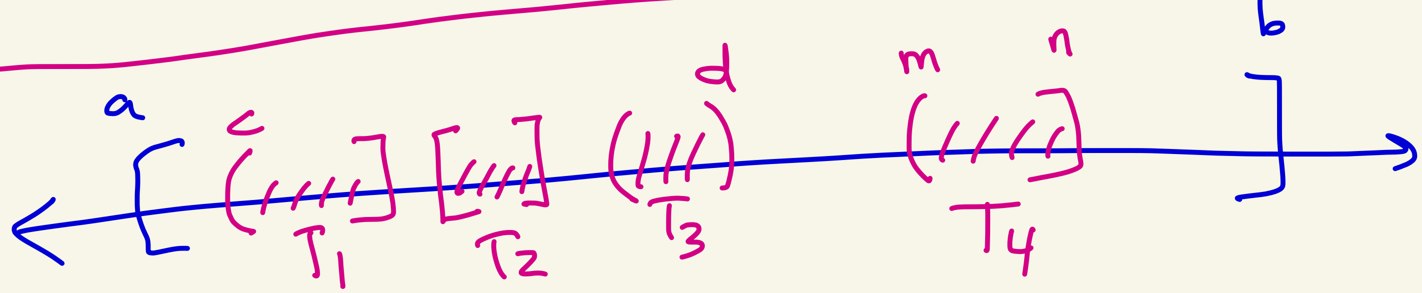
Thus, $\ell(T_1) = d-c \le b-a$.

$d \le b$
$-c \le -a$

Now suppose the problem is true for s disjoint bounded intervals.

Suppose we have $s+1$ disjoint bounded intervals $T_1, T_2, \ldots, T_s, T_{s+1}$ where $\bigcup_{i=1}^{s+1} T_i \subseteq [a,b]$.

Since the intervals are disjoint we can assume they are in order on the number line by reordering if necessary



picture for $s=3$, $s+1=4$

Let c be the left endpoint of T_1 and d be the right endpoint of T_s.

Then, $\bigcup_{i=1}^{s} T_i \subseteq [c,d]$ and by the induction hypothesis $\sum_{i=1}^{s} \ell(T_i) \leq d-c$.

Let m be the left endpoint of T_{s+1} and n be the right endpoint of T_{s+1}.

Note that $a \leq c \leq d \leq m \leq n \leq b$ and by construction $(d-c)+(n-m) \leq b-a$.

Thus,

$$\sum_{i=1}^{s+1} \ell(T_i) = \sum_{i=1}^{s} \ell(T_i) + \ell(T_{s+1})$$

$$\leq (d-c) + (n-m)$$

$$\leq b-a.$$

Thus, by induction the problem is proved.