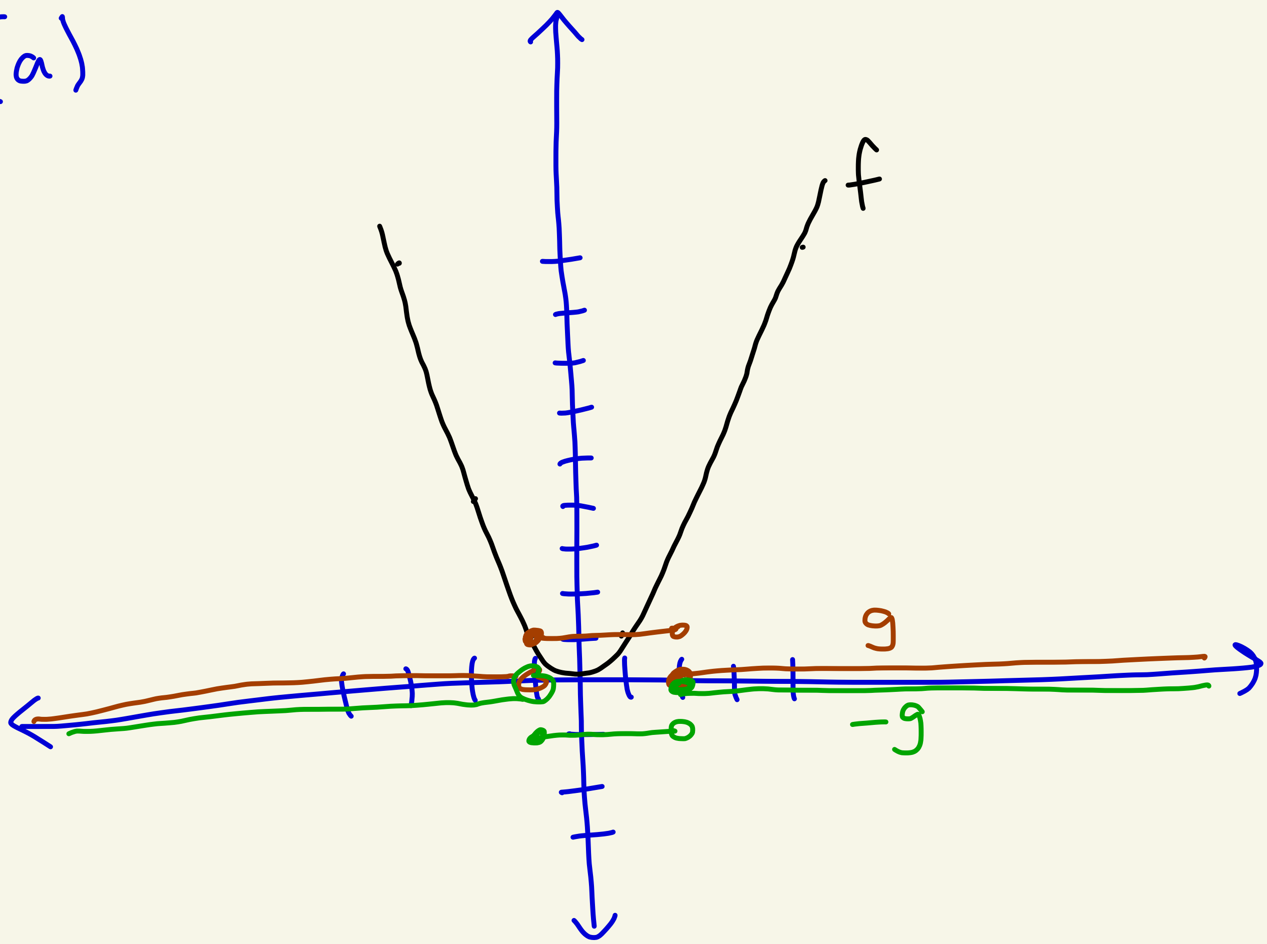
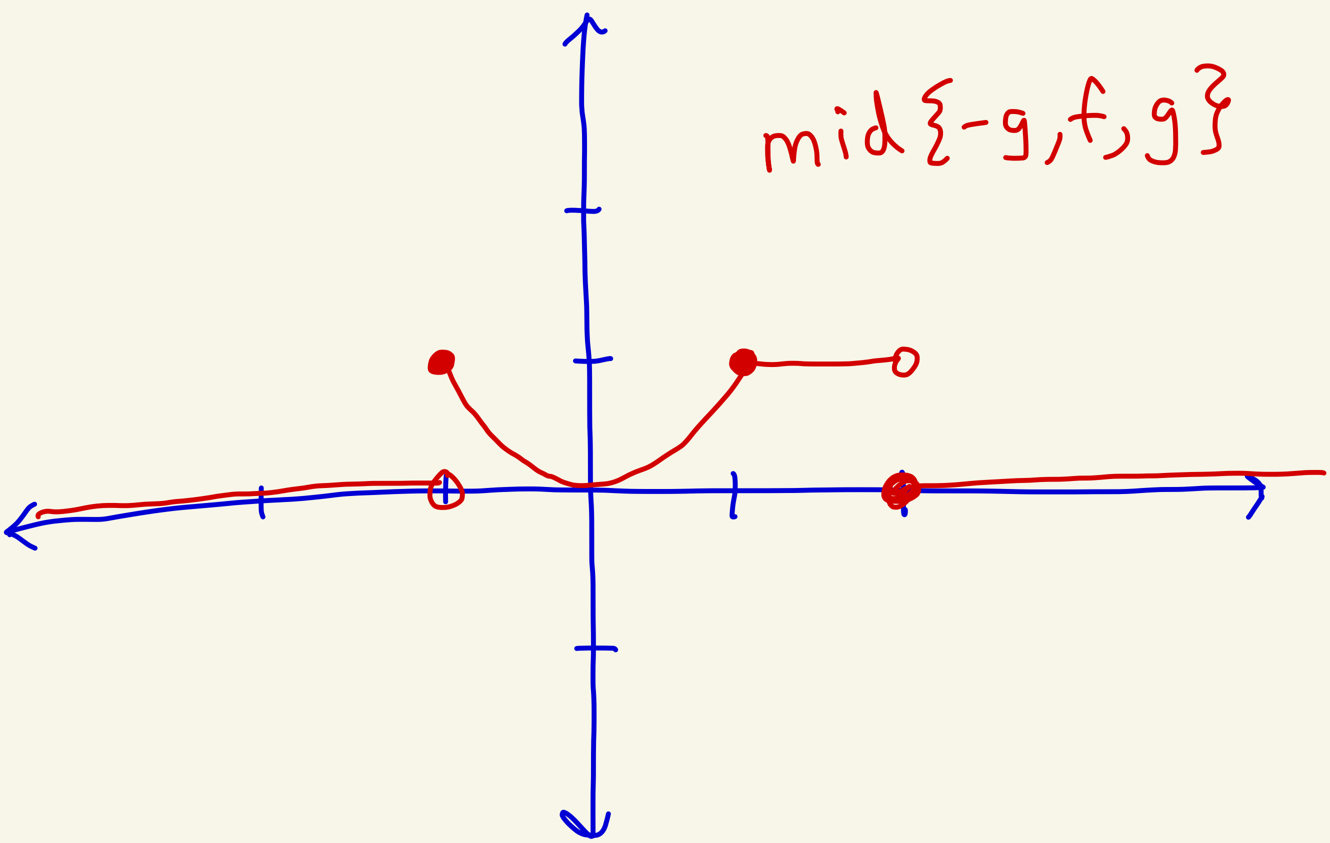
5800

HW 9 Solutions

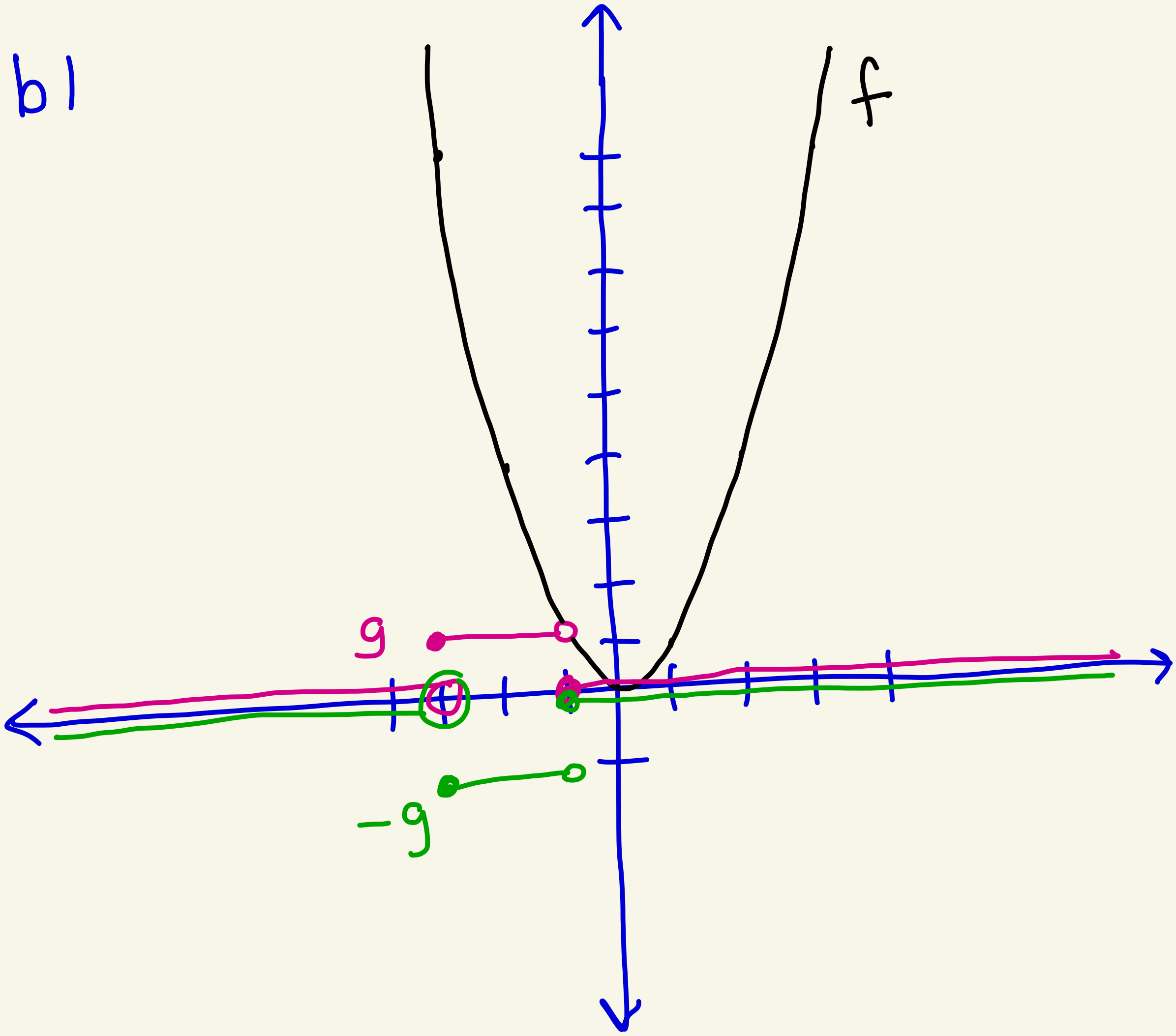
Measurable Functions

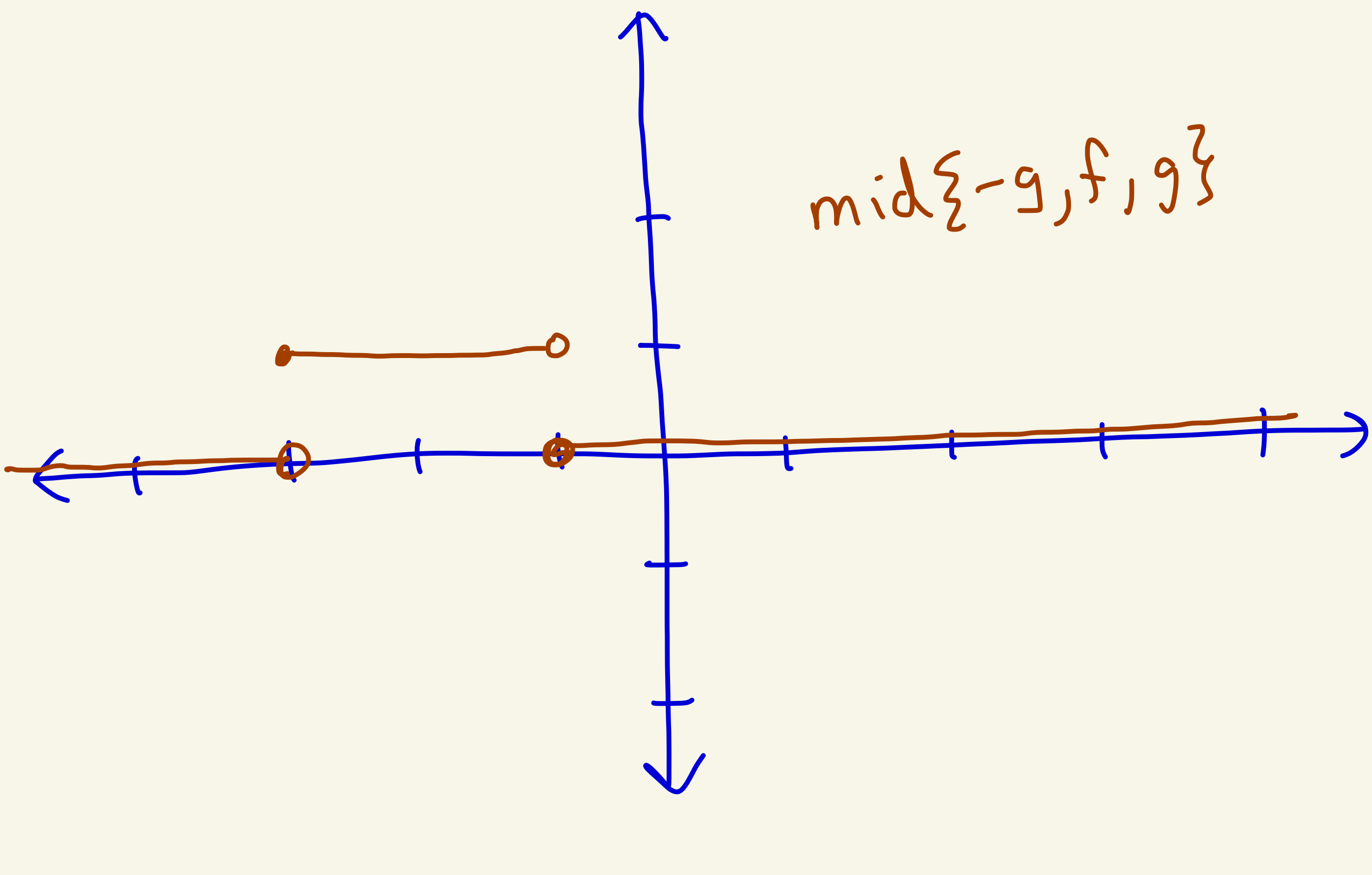
① (a)



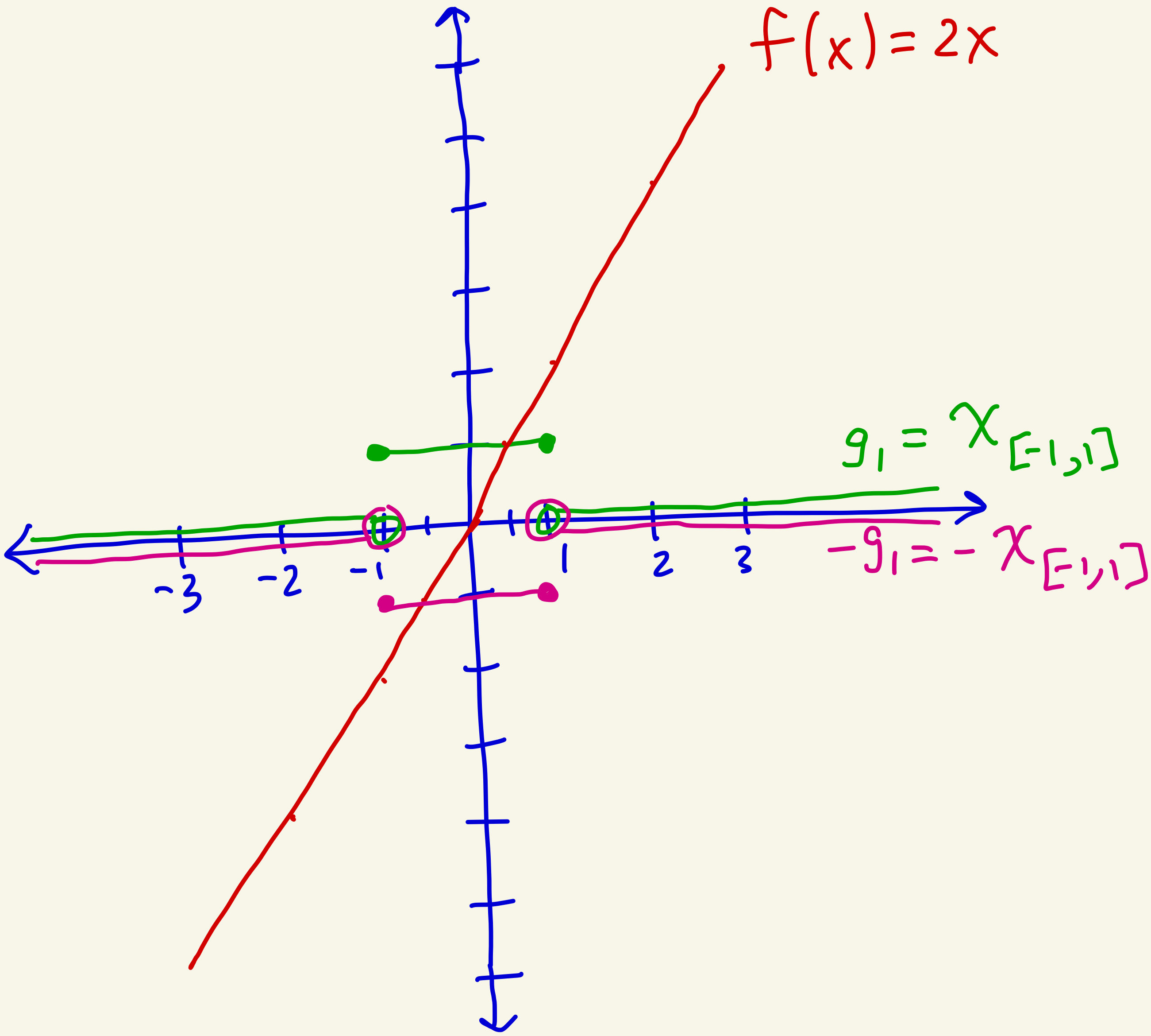


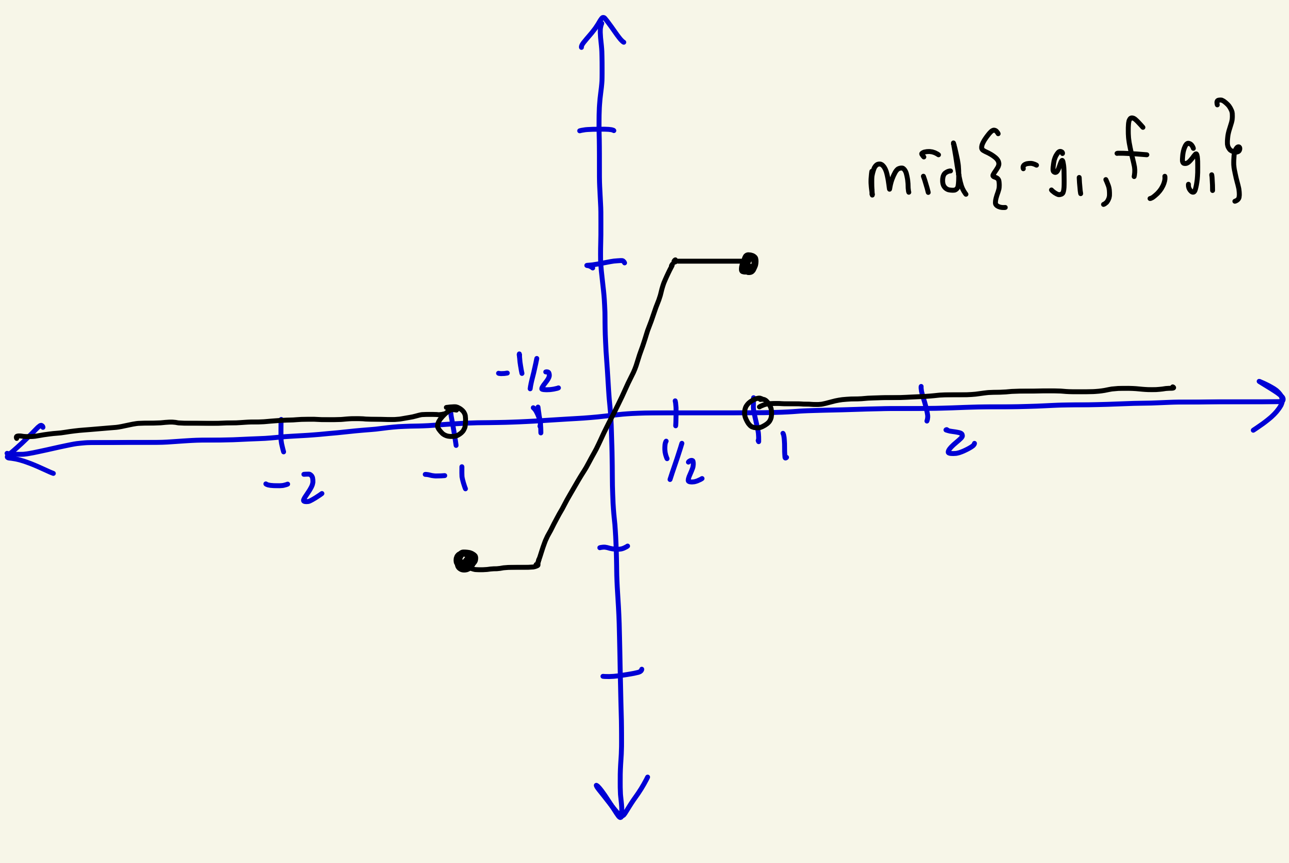
1 (b)

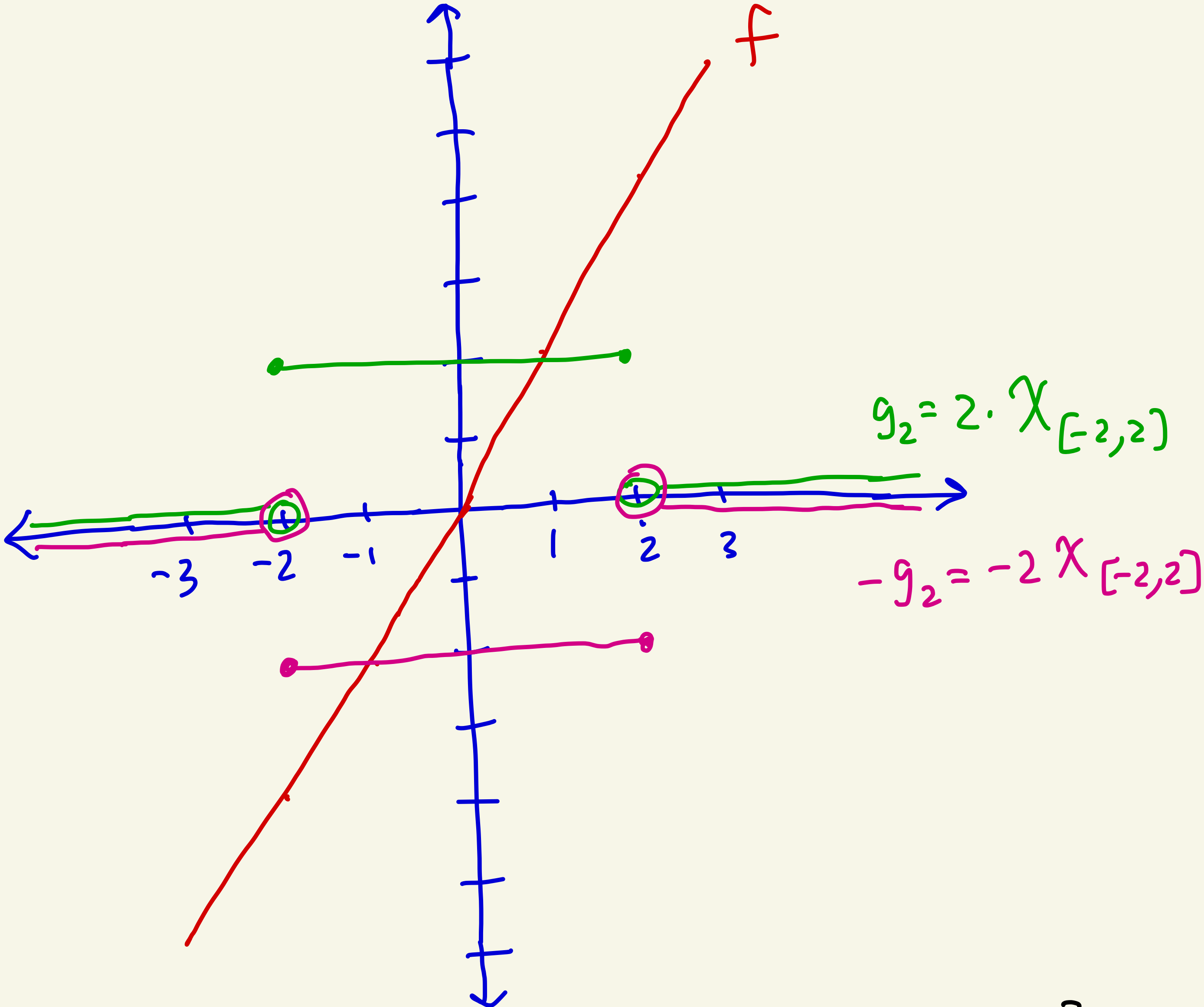




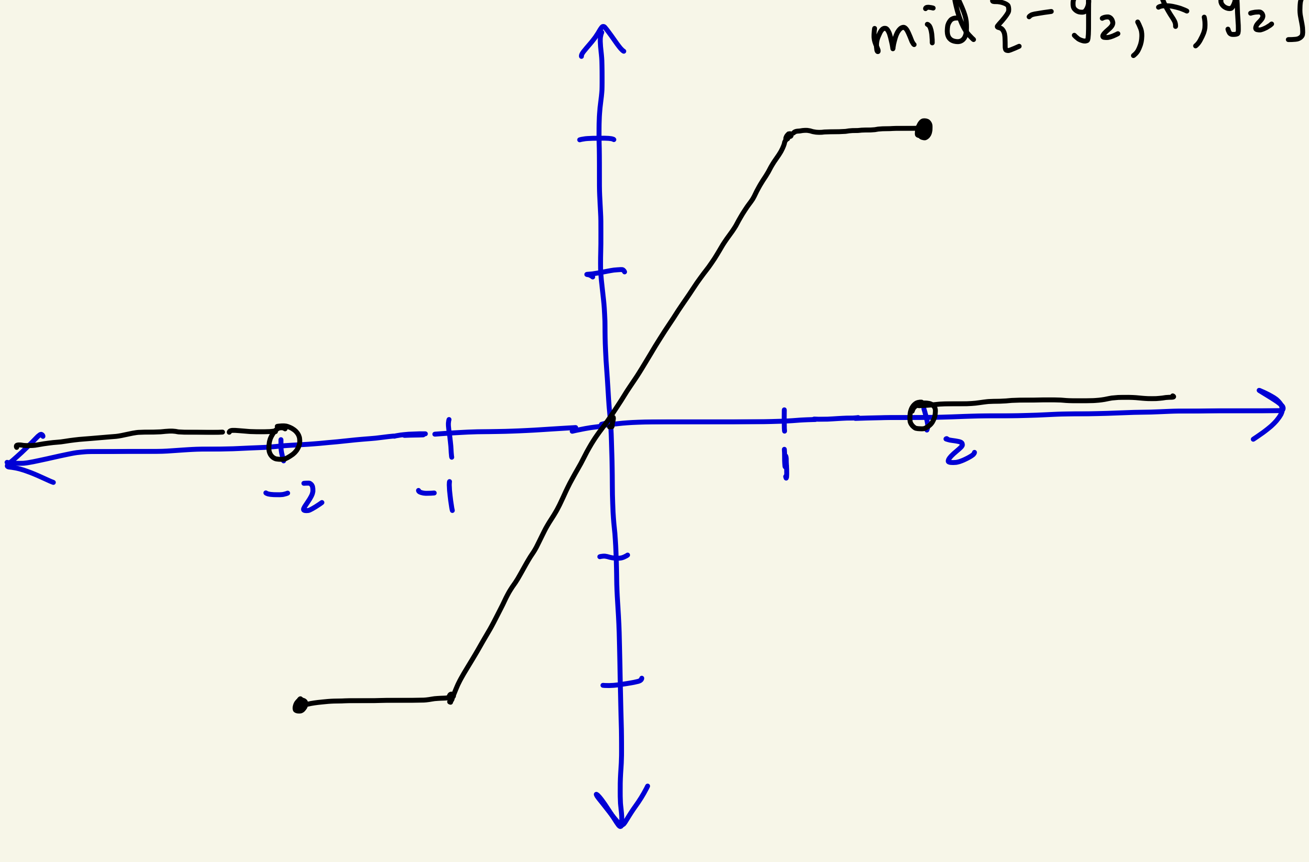
1 (c)





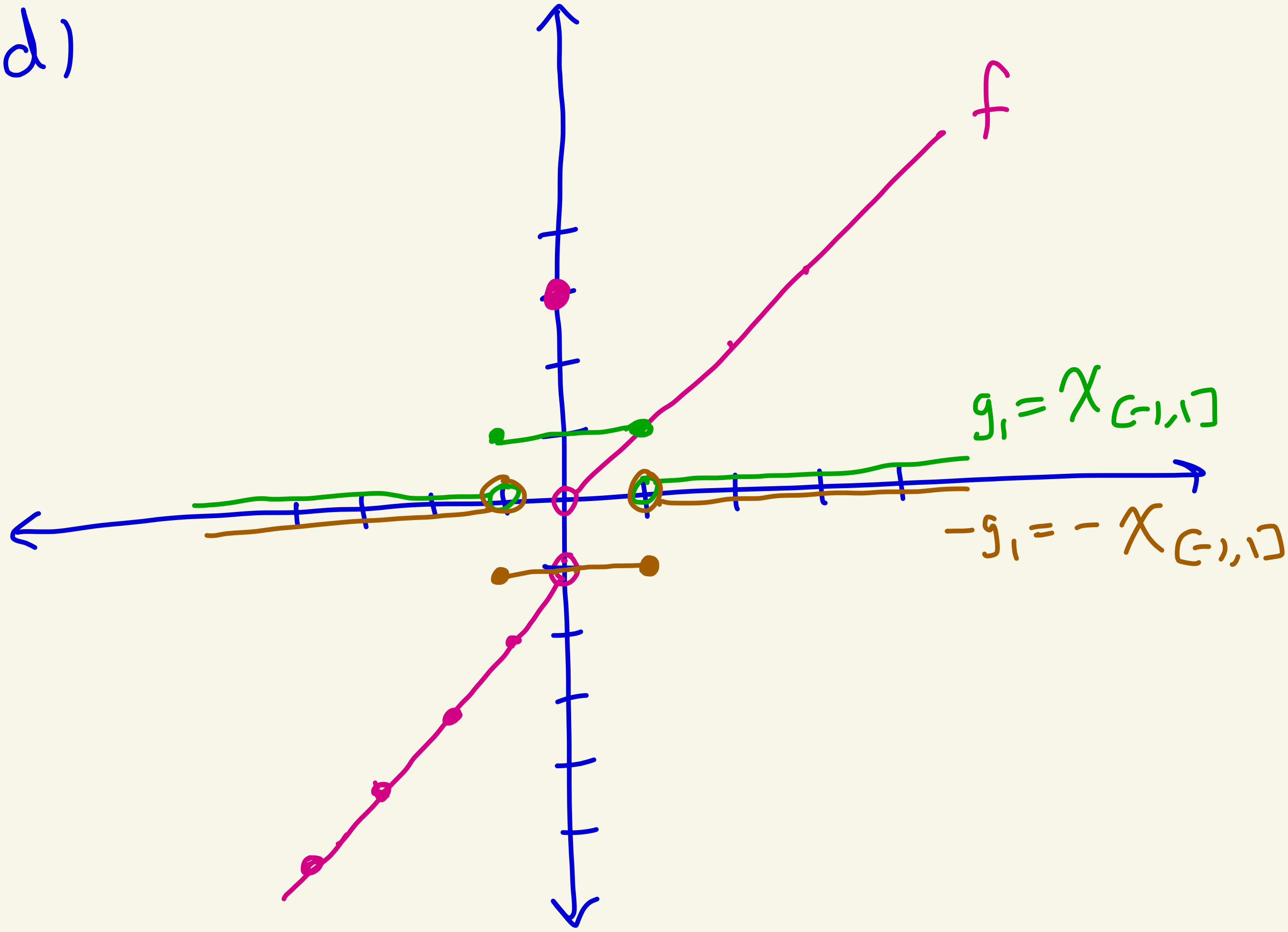


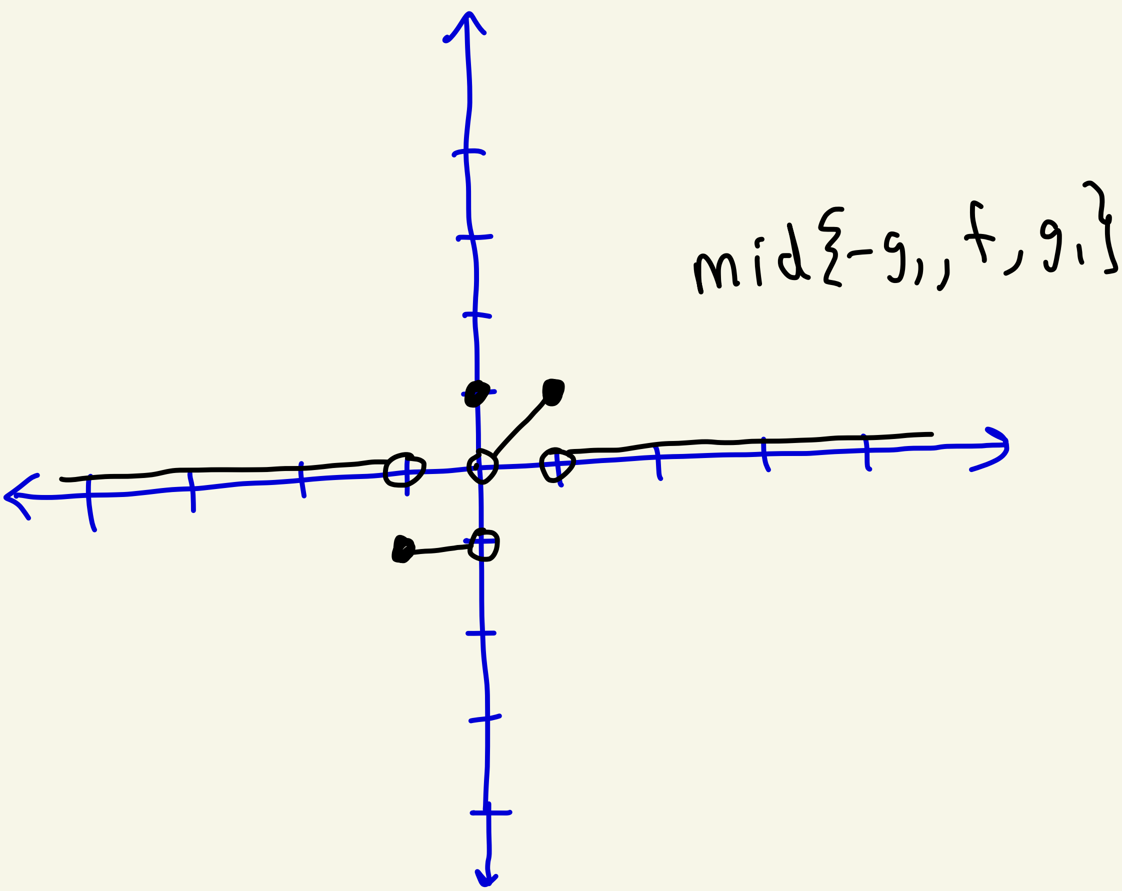
f
$g_2 = 2 \cdot \chi_{[-2,2]}$
$-g_2 = -2\chi_{[-2,2]}$
-3 -2 -1 1 2 3

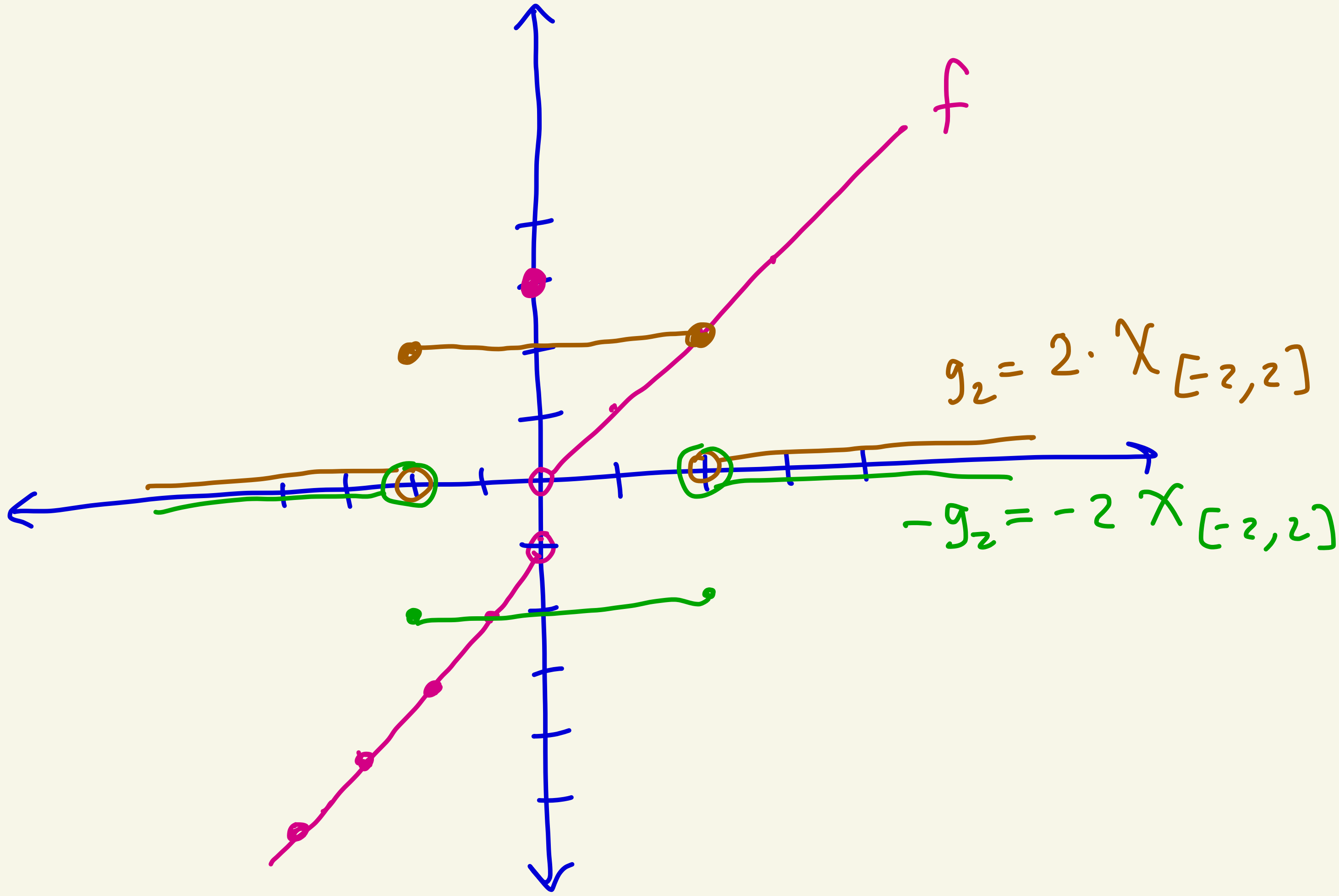


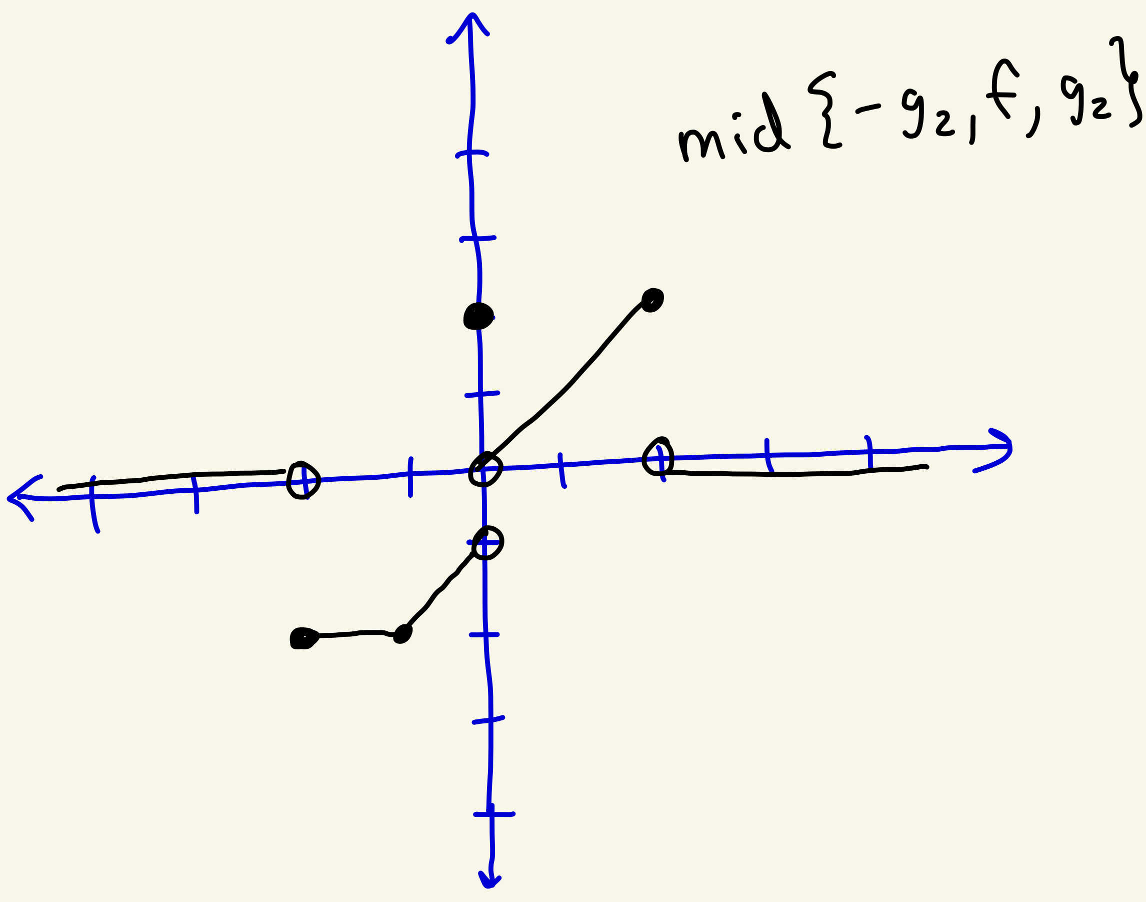
mid{$-g_2$, f, g_2}
-2 -1 1 2

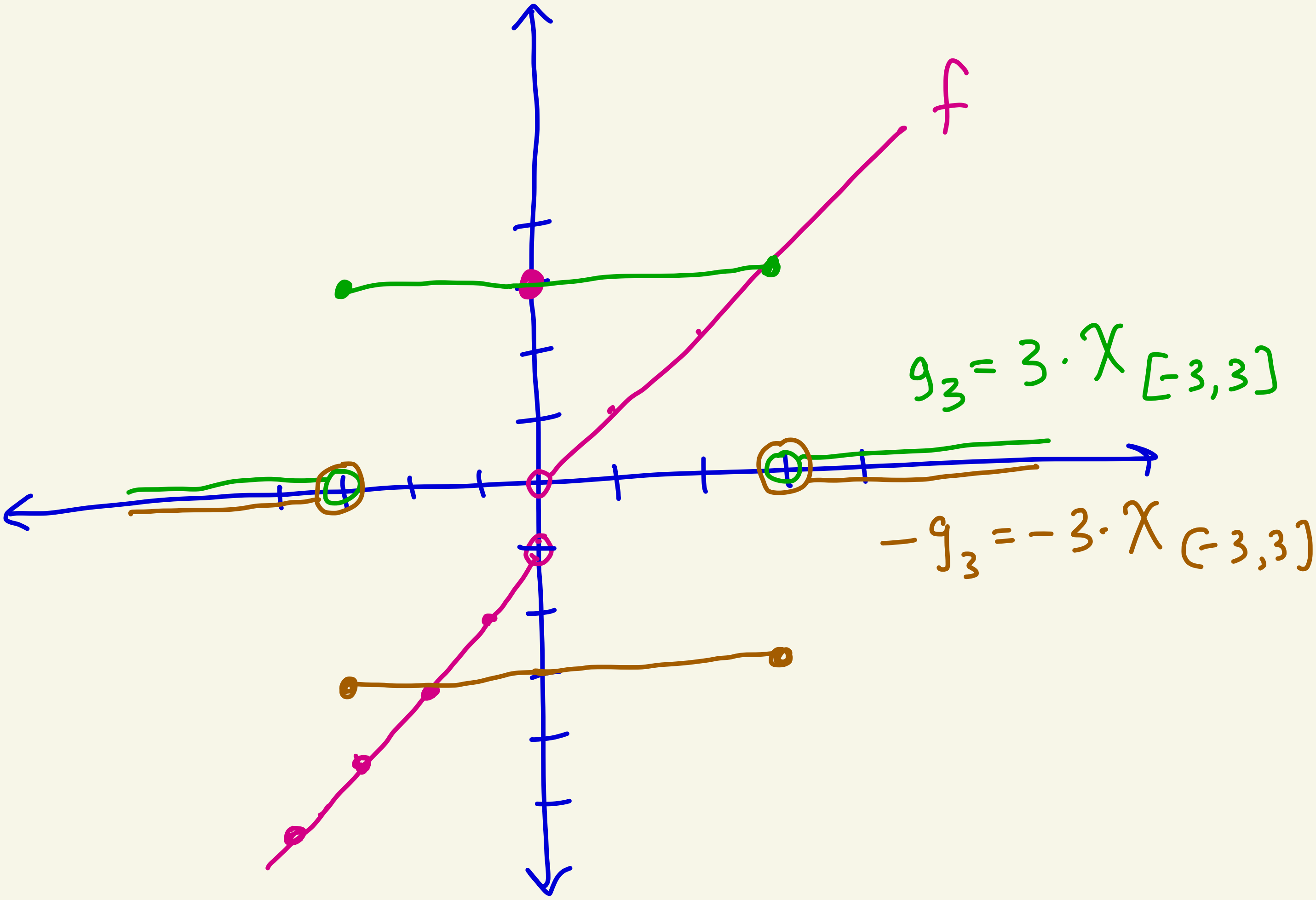
① (d)

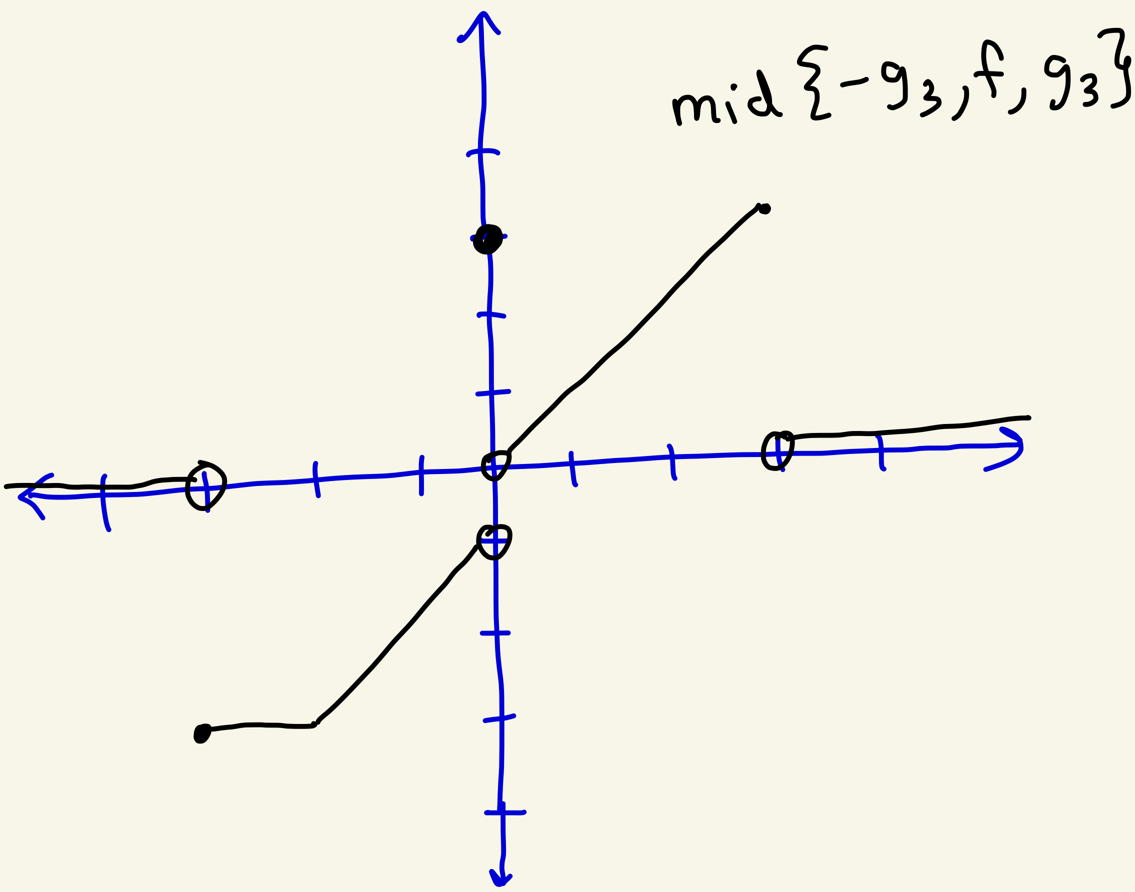


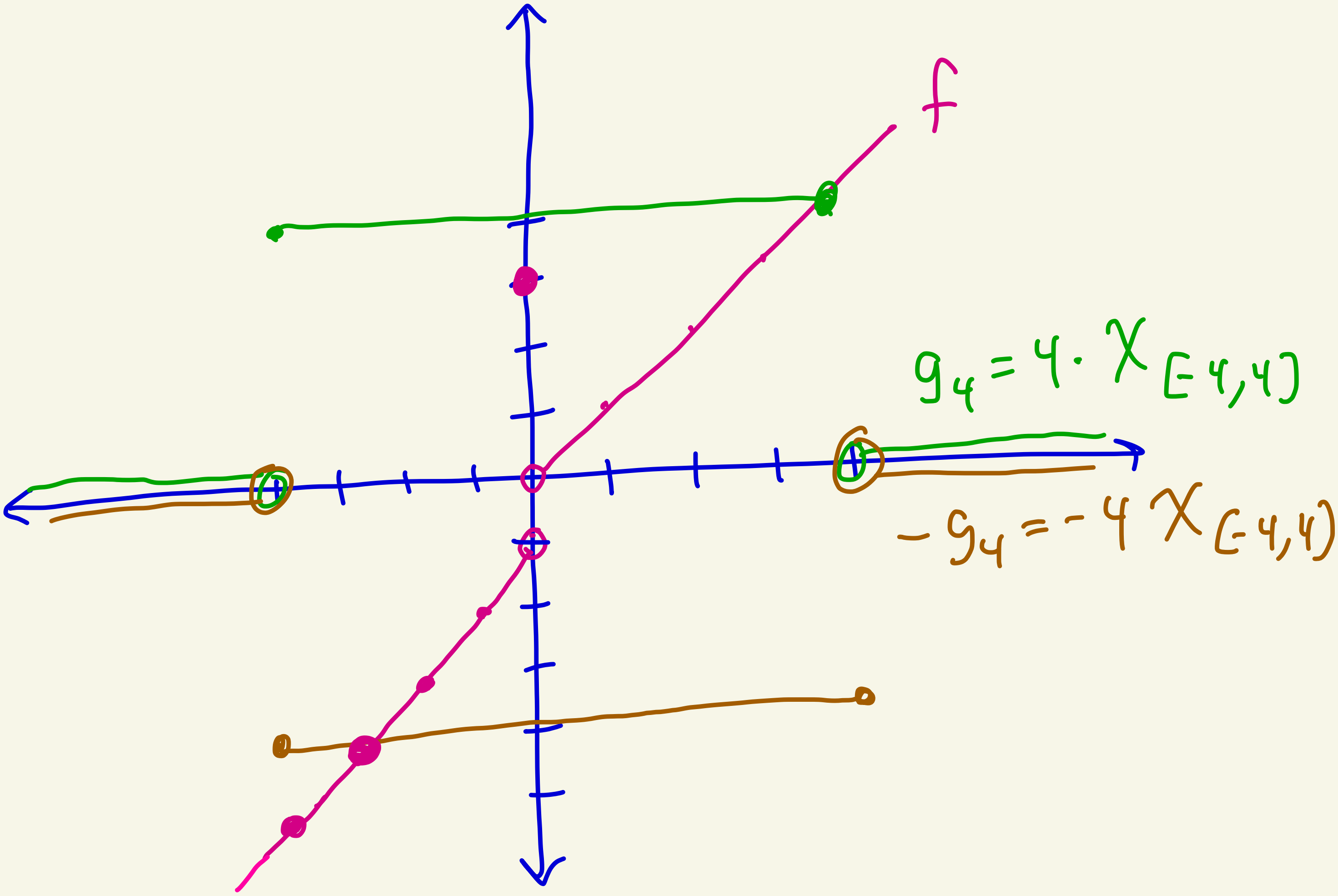



f
g2= 2 · X[-2,2]
-g2= -2 X[-2,2]

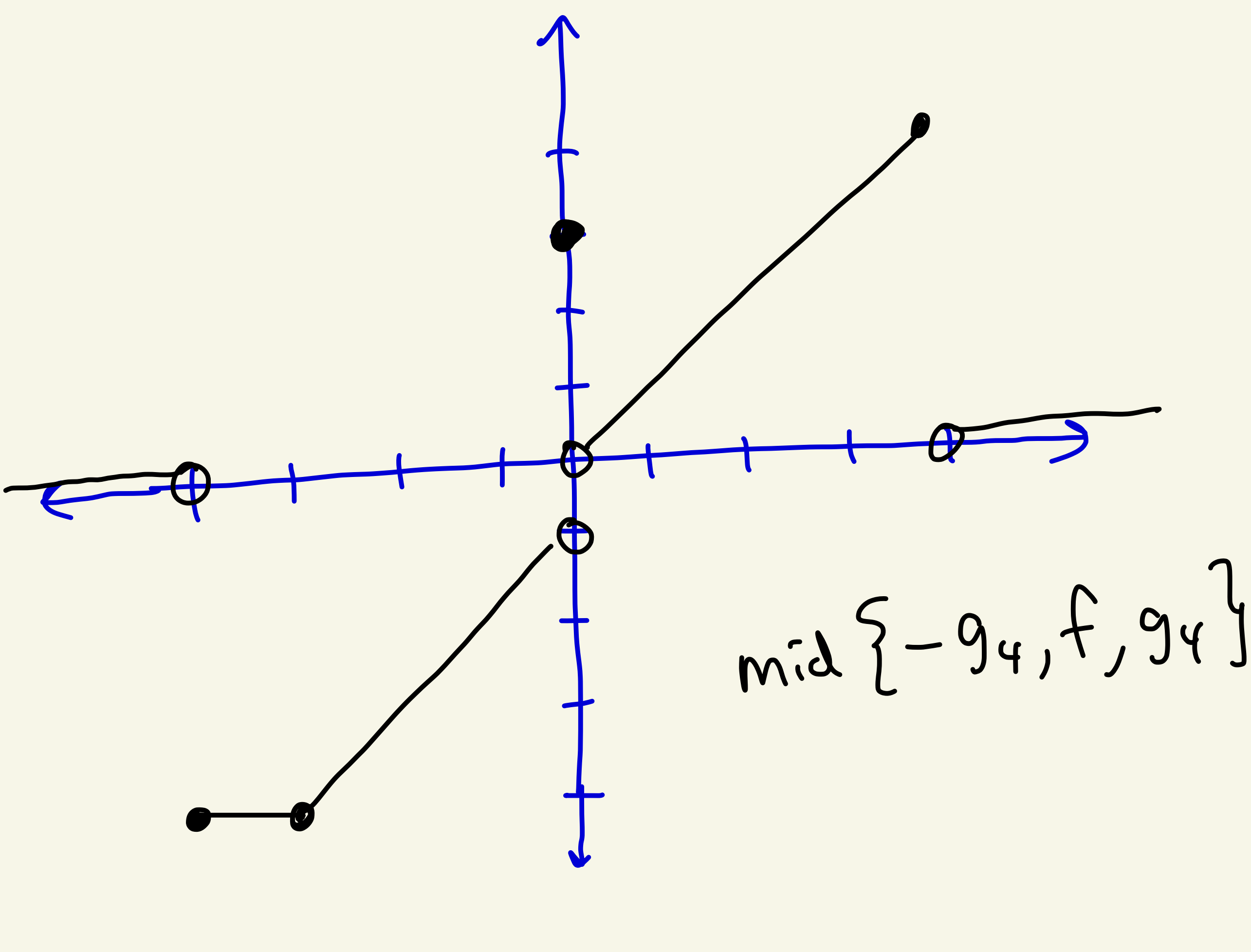

mid{-g2, f, g2}


f
g3 = 3 · χ[-3,3]
-g3 = -3 · χ[-3,3]


mid{-g3, f, g3}



f
$g_4 = 4 \cdot \chi_{[-4,4]}$
$-g_4 = -4\chi_{[-4,4]}$



$\text{mid}\{-g_4, f, g_4\}$

②

Let $x \in \mathbb{R}$ be fixed.

Pick $N_1 > 0$ large enough so that $-N_1 \leq f(x) \leq N_1$.

Pick $N_2 > 0$ large enough so that $-N_2 \leq x \leq N_2$.

Let $M = \max\{N_1, N_2\}$.

Then, $-M \leq f(x) \leq M$ and $-M \leq x \leq M$.

Thus,

$$-g_M(x) = -M \cdot \overbrace{\chi_{[-M,M]}(x)}^{1 \text{ since } -M \leq x \leq M}$$

$$= -M \leq f(x) \leq M$$

$$= M \cdot \underbrace{\chi_{[-M,M]}(x)}_{1} = g_M(x)$$

That is,

$$-g_M(x) \le f(x) \le g_M(x).$$

So,

$$f_M(x) = \text{mid}\{-g_M, f, g_M\}(x) = f(x).$$

Note that if $n \geq M$, then $[-M,M] \subseteq [-n,n]$ and so $\chi_{[-M,M]}(x) \le \chi_{[-n,n]}(x)$.

Thus, if $n \geq M$, then

$$-g_n(x) = -n \cdot \chi_{[-n,n]}(x) \le -n \cdot \chi_{[-M,M]}(x)$$

$$\le -M \cdot \chi_{[-M,M]}(x) = -g_M(x)$$

$$\le f(x) \le g_M(x) = M \cdot \chi_{[-M,M]}(x)$$

$$\le n \cdot \chi_{[-M,M]}(x) \le n \cdot \chi_{[-n,n]}(x)$$

$$= g_n(x).$$

That is, if $n \geq M$, then

$$-g_n(x) \leq f(x) \leq g_n(x).$$

So if $n \geq M$, then

$$f_n(x) = \text{mid}\{-g_n, f, g_n\}(x)$$

$$= \text{mid}\{-g_n(x), f(x), g_n(x)\} = f(x).$$

Thus given $\varepsilon > 0$, if $n \geq M$, then

$$|f_n(x) - f(x)| = |f(x) - f(x)|$$

$$= 0 < \varepsilon.$$

Therefore, $\lim_{n \to \infty} f_n(x) = f(x)$.

∎

③ Let $x \in \mathbb{R}$.

Since $h(x) = \operatorname{mid}\{-g(x), f(x), g(x)\}$ we may break the proof into three cases.

<u>Case 1:</u> Suppose $h(x) = -g(x)$.

Then, $|h(x)| = |-g(x)| = g(x)$. ← $g(x) \geq 0$

<u>Case 2:</u> Suppose $h(x) = f(x)$.

Then by the def of $\operatorname{mid}\{-g(x), f(x), g(x)\}$ we have that $-g(x) \leq f(x) \leq g(x)$.

Thus, $-g(x) \leq h(x) \leq g(x)$.

So, $|h(x)| \leq g(x)$.

<u>Case 3:</u> Suppose $h(x) = g(x)$.

Then, $|h(x)| = |g(x)| = g(x)$ ← $g(x) \geq 0$

In all three cases, $|h(x)| \leq g(x)$.

$\blacksquare$

④ (a)

From HW 4 problem 5, $\min\{\varphi_n, \psi_n\}$ is a step function for each $n \geq 1$ since φ_n and ψ_n are step functions.

We now show that the sequence $\min\{\varphi_n, \psi_n\}$ is non-decreasing.

Let $n \geq 1$ be fixed.

Let $x \in \mathbb{R}$.

Since $(\varphi_n)_{n=1}^{\infty}$ is non-decreasing we have that $\boxed{\varphi_n(x) \leq \varphi_{n+1}(x).}$

Since $(\psi_n)_{n=1}^{\infty}$ is non-decreasing we have that $\boxed{\psi_n(x) \leq \psi_{n+1}(x).}$

We break the rest of the proof into 4 cases.

↓

Case 1: Suppose $\min\{\varphi_n(x), \psi_n(x)\} = \varphi_n(x)$ and $\min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \varphi_{n+1}(x)$.

Then,

$$\begin{aligned}\min\{\varphi_n, \psi_n\}(x) &= \min\{\varphi_n(x), \psi_n(x)\} \\ &= \varphi_n(x) \leq \varphi_{n+1}(x) \\ &= \min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\ &= \min\{\varphi_{n+1}, \psi_{n+1}\}(x)\end{aligned}$$

Case 2: Suppose $\min\{\varphi_n(x), \psi_n(x)\} = \varphi_n(x)$ and $\min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \psi_{n+1}(x)$.

Since $\min\{\varphi_n(x), \psi_n(x)\} = \varphi_n(x)$ we know that $\varphi_n(x) \leq \psi_n(x)$.

Then,

$$\begin{aligned}\min\{\varphi_n, \psi_n\}(x) &= \min\{\varphi_n(x), \psi_n(x)\} \\ &= \varphi_n(x) \leq \psi_n(x) \leq \psi_{n+1}(x) \\ &= \min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\ &= \min\{\varphi_{n+1}, \psi_{n+1}\}(x)\end{aligned}$$

Case 3: Suppose $\min\{\varphi_n(x), \psi_n(x)\} = \psi_n(x)$ and $\min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \varphi_{n+1}(x)$.

Since $\min\{\varphi_n(x), \psi_n(x)\} = \psi_n(x)$ we know that $\psi_n(x) \le \varphi_n(x)$

Then,

$$\begin{aligned}\min\{\varphi_n, \psi_n\}(x) &= \min\{\varphi_n(x), \psi_n(x)\} \\ &= \psi_n(x) \le \varphi_n(x) \le \varphi_{n+1}(x) \\ &= \min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\ &= \min\{\varphi_{n+1}, \psi_{n+1}\}(x)\end{aligned}$$

Case 4: Suppose $\min\{\varphi_n(x), \psi_n(x)\} = \psi_n(x)$ and $\min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \psi_{n+1}(x)$.

Then,

$$\begin{aligned}\min\{\varphi_n, \psi_n\}(x) &= \min\{\varphi_n(x), \psi_n(x)\} \\ &= \psi_n(x) \le \psi_{n+1}(x) \\ &= \min\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\ &= \min\{\varphi_{n+1}, \psi_{n+1}\}(x)\end{aligned}$$

From cases 1-4 we get that $\left(\min\{\varphi_n, \psi_n\}\right)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions.

④ (b)

From HW 4 problem 5, $\max\{\varphi_n, \psi_n\}$ is a step function for each $n \geq 1$ since φ_n and ψ_n are step functions.

We now show that the sequence $\max\{\varphi_n, \psi_n\}$ is non-decreasing.

Let $n \geq 1$ be fixed.

Let $x \in \mathbb{R}$.

Since $(\varphi_n)_{n=1}^{\infty}$ is non-decreasing we have that $\boxed{\varphi_n(x) \leq \varphi_{n+1}(x).}$

Since $(\psi_n)_{n=1}^{\infty}$ is non-decreasing we have that $\boxed{\psi_n(x) \leq \psi_{n+1}(x).}$

We break the rest of the proof into 4 cases. ↓

Case 1: Suppose $\max\{\varphi_n(x), \psi_n(x)\} = \varphi_n(x)$ and $\max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \varphi_{n+1}(x)$.

Then,

$$\begin{aligned}
\max\{\varphi_n, \psi_n\}(x) &= \max\{\varphi_n(x), \psi_n(x)\} \\
&= \varphi_n(x) \leq \varphi_{n+1}(x) \\
&= \max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\
&= \max\{\varphi_{n+1}, \psi_{n+1}\}(x)
\end{aligned}$$

Case 2: Suppose $\max\{\varphi_n(x), \psi_n(x)\} = \varphi_n(x)$ and $\max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \psi_{n+1}(x)$.

Since $\max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \psi_{n+1}(x)$ we know that $\varphi_{n+1}(x) \leq \psi_{n+1}(x)$

Then,

$$\begin{aligned}
\max\{\varphi_n, \psi_n\}(x) &= \max\{\varphi_n(x), \psi_n(x)\} \\
&= \varphi_n(x) \leq \varphi_{n+1}(x) \leq \psi_{n+1}(x) \\
&= \max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\
&= \max\{\varphi_{n+1}, \psi_{n+1}\}(x)
\end{aligned}$$

<u>Case 3:</u> Suppose $\max\{\varphi_n(x), \psi_n(x)\} = \psi_n(x)$ and $\max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \varphi_{n+1}(x)$.

Since $\max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \varphi_{n+1}(x)$ we know that $\psi_{n+1}(x) \leq \varphi_{n+1}(x)$.

Then,

$$\begin{aligned}\max\{\varphi_n, \psi_n\}(x) &= \max\{\varphi_n(x), \psi_n(x)\} \\ &= \psi_n(x) \leq \psi_{n+1}(x) \leq \varphi_{n+1}(x) \\ &= \max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\ &= \max\{\varphi_{n+1}, \psi_{n+1}\}(x)\end{aligned}$$

<u>Case 4:</u> Suppose $\max\{\varphi_n(x), \psi_n(x)\} = \psi_n(x)$ and $\max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} = \psi_{n+1}(x)$.

Then,

$$\begin{aligned}\max\{\varphi_n, \psi_n\}(x) &= \max\{\varphi_n(x), \psi_n(x)\} \\ &= \psi_n(x) \leq \psi_{n+1}(x) \\ &= \max\{\varphi_{n+1}(x), \psi_{n+1}(x)\} \\ &= \max\{\varphi_{n+1}, \psi_{n+1}\}(x)\end{aligned}$$

From cases 1–4 we get that $\left(\max\{\varphi_n, \psi_n\}\right)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions.

⑤ (a) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be in L^0.

Then there exist non-decreasing sequences of step functions $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ such that $\varphi_n \to f$ on an almost everywhere set A_1 and $\psi_n \to g$ on an almost everywhere set A_2.

Furthermore, $\lim_{n\to\infty} \int \varphi_n$ converges to $\int f$ and $\lim_{n\to\infty} \int \psi_n$ converges to $\int g$.

From HW 3, $A_1 \cap A_2$ is an almost everywhere set.

And both $\lim_{x\to\infty} \varphi_n(x) = f(x)$ and $\lim_{x\to\infty} \psi_n(x) = g(x)$ for all $x \in A_1 \cap A_2$

Consider the sequences
$(\max\{\varphi_n, \psi_n\})_{n=1}^{\infty}$ and $(\min\{\varphi_n, \psi_n\})_{n=1}^{\infty}$

From the previous HW problem these are both non-decreasing sequences of step functions.

From HW 2, problem 3, we have that $\min\{\varphi_n, \psi_n\} \longrightarrow \min\{f, g\}$ on $A_1 \cap A_2$ and
$\max\{\varphi_n, \psi_n\} \longrightarrow \max\{f, g\}$ on $A_1 \cap A_2$.

To get that $\max\{f, g\}$ and $\min\{f, g\}$ are in L^0 we just have to bound the sequences
$\left(\int \min\{\varphi_n, \psi_n\}\right)_{n=1}^{\infty}$ and $\left(\int \max\{\varphi_n, \psi_n\}\right)_{n=1}^{\infty}$.

Note that for all $x \in \mathbb{R}$ we have that both

$$\min\{\varphi_n(x), \psi_n(x)\} \leq \varphi_n(x) + \psi_n(x)$$

and

$$\max\{\varphi_n(x), \psi_n(x)\} \leq \varphi_n(x) + \psi_n(x)$$

Thus, $\int \min\{\varphi_n, \psi_n\} \leq \int \varphi_n + \psi_n$

and $\int \max\{\varphi_n, \psi_n\} \leq \int \varphi_n + \psi_n$

So we just need to bound the sequence $\left(\int (\varphi_n + \psi_n) \right)_{n=1}^{\infty}$.

This sequence is bounded because it converges since $\lim_{n \to \infty} \int (\varphi_n + \psi_n) =$

$$= \lim_{n \to \infty} \int \varphi_n + \lim_{n \to \infty} \int \psi_n = \int f + \int g.$$

∎

(5) (b) Let $f \in L^1$.

Then $f = g - h$ where $g, h \in L^0$.

<u>Claim:</u> $|f| = \max\{g,h\} - \min\{g,h\}$.

<u>pf of claim:</u>

Let $x \in \mathbb{R}$.

<u>case 1:</u> Suppose $g(x) \leq h(x)$.

Then, $g(x) - h(x) \leq 0$.

So,

$$|f(x)| = |g(x) - h(x)| = -(g(x) - h(x))$$
$$= h(x) - g(x)$$

and

$$\max\{g,h\}(x) - \min\{g,h\}(x)$$
$$= \max\{g(x), h(x)\} - \min\{g(x), h(x)\}$$
$$= \max\{g(x)$$
$$= h(x) - g(x)$$

<u>Case 2:</u> Suppose $h(x) < g(x)$.
Then, $g(x) - h(x) > 0$.

So,
$$|f(x)| = |g(x) - h(x)| = g(x) - h(x)$$

and
$$\max\{g,h\}(x) - \min\{g,h\}(x)$$
$$= \max\{g(x), h(x)\} - \min\{g(x), h(x)\}$$
$$= g(x) - h(x).$$

Thus in either case we have that $|f| = \max\{g,h\} - \min\{g,h\}$

By part (a), since $g, h \in L^0$ we have that $\max\{g,h\}$ and $\min\{g,h\}$ are in L^0.

Thus, $|f| = \max\{g,h\} - \min\{g,h\} \in L^1$

$\blacksquare$

(5) (c) Let $f, g \in L^1$.

Then, $f - g \in L^1$.

By part (b) we have that $|f-g| \in L^1$.

As in HW 4 problem 5 one can show that

$$\min\{f,g\} = \frac{1}{2}f + \frac{1}{2}g - \frac{1}{2}|f-g|.$$

See HW problem 5 solutions for how to prove this

Since L^1 is closed under addition, subtraction, and multiplying by a real number we get that $\min\{f,g\} \in L^1$.

A similar proof using

$$\max\{f,g\} = \frac{1}{2}f + \frac{1}{2}g + \frac{1}{2}|f-g|$$

shows that $\max\{f,g\} \in L^1$.

(6) Since $b \geq 0$ we have that $-b \leq b$. Thus, there are three possibilities:
$a < -b \leq b$ or $-b \leq a \leq b$ or $-b \leq b < a$.

Thus,

$$\text{mid}\{-b, a, b\} = \begin{cases} -b & \text{if } a < -b \leq b \\ a & \text{if } -b \leq a \leq b \\ b & \text{if } -b \leq b < a \end{cases}$$

$$= \begin{cases} -b & \text{if } a < -b \\ a & \text{if } -b \leq a \leq b \\ b & \text{if } b < a \end{cases}$$

This gives part of the result.

Let's now show that

$$\text{mid}\{-b, a, b\} = \max\{-b, \min\{a, b\}\}.$$

If $a < -b \leq b$, then

$$\max\{-b, \min\{a, b\}\} = \max\{-b, a\} = -b = \text{mid}\{-b, a, b\}.$$

If $-b \leq a \leq b$, then

$$\max\{-b, \min\{a, b\}\} = \max\{-b, a\} = a = \text{mid}\{-b, a, b\}$$

If $-b \le b \le a$, then

$$\max\{-b, \min\{a,b\}\} = \max\{-b, b\} = b$$
$$= \mathrm{mid}\{-b, a, b\}$$

In all three cases we have that

$$\max\{-b, \mathrm{mid}\{a,b\}\} = \mathrm{mid}\{-b, a, b\}.$$

∎

7 (a)

By the previous HW problem

$$\text{mid}\{-b_n, a_n, b_n\} = \max\{-b_n, \min\{a_n, b_n\}\}.$$

Since $a_n \to a$ and $b_n \to b$, by HW 2 problem 3, $\min\{a_n, b_n\} \to \min\{a, b\}$.

Since $-b_n \to -b$ and $\min\{a_n, b_n\} \to \min\{a, b\}$ by HW 2 problem 3,

$$\max\{-b_n, \min\{a_n, b_n\}\} \longrightarrow \max\{-b, \min\{a, b\}\}.$$

Thus, by the previous HW problem,

$$\text{mid}\{-b_n, a_n, b_n\} \longrightarrow \text{mid}\{-b, a, b\}$$

∎

7 (b)

We have that $\lim_{n\to\infty} f_n(x) = f(x)$

for all $x \in A$ where A is an almost everywhere set.

By problem 7(a) since $\lim_{n\to\infty} g(x) = g(x)$ for all x we have that

$$\lim_{n\to\infty} \text{mid}\{-g(x), f_n(x), g(x)\}$$

$$= \text{mid}\{-g(x), f(x), g(x)\}$$

for all $x \in A$.

This proves the result. ☒

⑧ (a)

Let f and h be measurable.
Then from a theorem in class, there exist sequences $(f_n)_{n=1}^{\infty}$ and $(h_n)_{n=1}^{\infty}$ where $f_n \in L^1$ and $h_n \in L^1$ for all $n \geqslant 1$ such that $h_n \to h$ almost everywhere and $f_n \to f$ almost everywhere.

Since $f_n \in L^1$ and $h_n \in L^1$ for all $n \geqslant 1$, we have that $f_n + h_n \in L^1$ for all $n \geqslant 1$.

By HW 6 problem 6, $f_n + h_n \to f + h$ almost everywhere.

Thus, $(f_n + h_n)_{n=1}^{\infty}$ is a sequence of L^1 functions with $f_n + h_n \to f + h$ for almost all x.

By a theorem in class, $f + h$ is measurable. ▨

(8) (b) Since f is measurable, there exists a sequence $(f_n)_{n=1}^{\infty}$ of L^1 functions where $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$ where A is an almost everywhere set.

Then, $\lim_{n\to\infty} \alpha f_n(x) = \alpha \lim_{n\to\infty} f_n(x)$

$= \alpha f(x)$

for all $x \in A$.

From a theorem from class, since $f_n \in L^1$ we know that $\alpha f_n \in L^1$.

Thus, $(\alpha f_n)_{n=1}^{\infty}$ is a sequence of L^1 functions that converges almost everywhere to αf.

By a theorem from class, αf is measurable. ▨

(8) (c) Since f and h are measurable there exists sequences $(f_n)_{n=1}^{\infty}$ and $(h_n)_{n=1}^{\infty}$ of L^1 functions where $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A_1$ and $\lim_{n\to\infty} h_n(x) = h(x)$ for all $x \in A_2$ where A_1 and A_2 are almost everywhere sets.

Thus, $A_1 \cap A_2$ is an almost everywhere set from a theorem from class and HW 3.

From HW 2 problem 3, if $x \in A_1 \cap A_2$ we have that

$$\lim_{n\to\infty} \min\{f_n(x), h_n(x)\} = \min\{f(x), h(x)\}$$

for all $x \in A_1 \cap A_2$.

From exercise 5 above,

$\min\{f_n, h_n\} \in L^1$ for all $n \geq 1$

because $f_n, h_n \in L^1$ for all $n \geq 1$.

Thus, $\left(\min\{f_n, h_n\}\right)_{n=1}^{\infty}$ is a sequence of L^1 functions that converges almost everywhere to $\min\{f, h\}$.

So, by a theorem in class, $\min\{f, h\}$ is measurable.

$\blacksquare$

⑧ (d)

Do the same proof as 8(c)
but replace min by max.

(8) (e) Let f be a measurable function. Let g be a non-negative function with $g \in L^1$.

Suppose $|f(x)| \le g(x)$ for all $x \in A$ where A is an almost everywhere set.

Since $|f(x)| \le g(x)$ for all $x \in A$ we have that $-g(x) \le f(x) \le g(x)$ for all $x \in A$.

Thus, $\text{mid}\{-g(x), f(x), g(x)\} = f(x)$ for all $x \in A$.

Because f is measurable and $g \in L^1$ and $g \geq 0$ we know $\text{mid}\{-g, f, g\} \in L^1$.

Since $\text{mid}\{-g, f, g\} \in L^1$ and

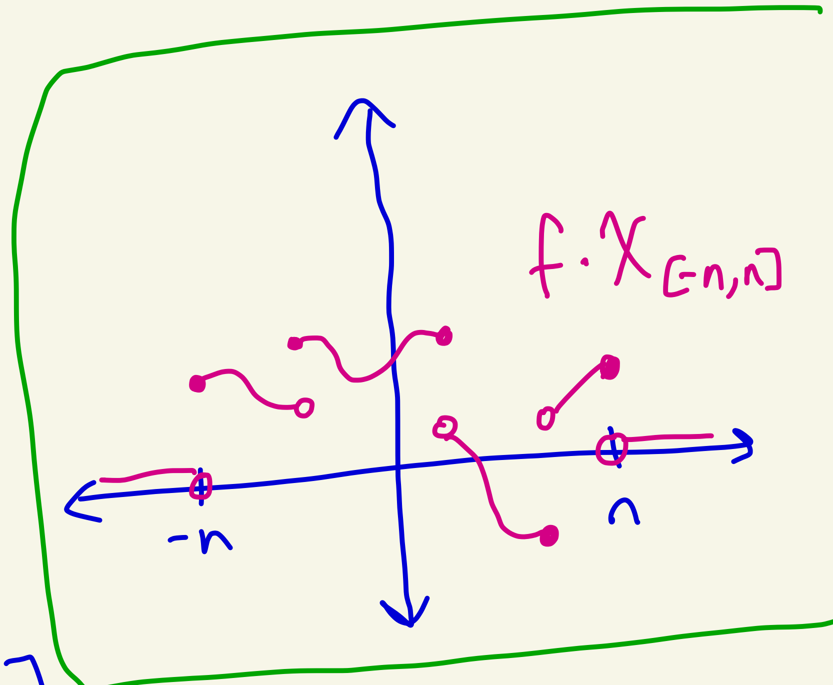
$f = \text{mid}\{-g, f, g\}$ almost everywhere

we know that $f \in L^1$. $\blacksquare$

⑨ Given $n \geq 1$, let $f_n = f \cdot \chi_{[-n,n]}$.

Thus,

$$f_n(x) = \begin{cases} f(x) & \text{when } -n \leq x \leq n \\ 0 & \text{otherwise} \end{cases}$$



Claim: $f_n \in L^1$ for $n \geq 1$

Let $F = E \cap (-n,n)$

Then F has measure zero.

Since f_n is continuous on $(-n,n) - F$ and f_n is bounded on $[-n,n]$ and f_n vanishes outside $[-n,n]$, we have that $f_n \in L^1$ [By Topic 8 Theorem]

Claim: $\lim_{n \to \infty} f_n(x) = f(x)$

Fix $x \in \mathbb{R}$. Fix $\varepsilon > 0$.

Pick $N > 0$ where $-N < x < N$.

Then, if $n \geq N$ we have that

$$-n \leq -N \leq x \leq N \leq n.$$

So, if $n \geq N$, then

$$f_n(x) = f(x) \cdot \underbrace{\chi_{[-n,n]}(x)}_{1} = f(x)$$

So, if $n \geq N$, then

$$|f_n(x) - f(x)| = |f(x) - f(x)|$$
$$= 0 < \varepsilon.$$

So, $\lim_{n \to \infty} f_n(x) = f(x)$ claim

Thus, $(f_n)_{n=1}^{\infty}$ is a sequence of L^1 functions converging to f on all of $\mathbb{R}$. Thus, by a theorem from class f is a measurable function. ☐