On $L(d,1)$-labelings of graphs

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Abstract

Given a graph G and a positive integer d, an $L(d,1)$-labeling of G is a function f that assigns to each vertex of G a non-negative integer such that if two vertices u and v are adjacent, then $|f(u)-f(v)| \geq d$; if u and v are not adjacent but there is a two-edge path between them, then $|f(u)-f(v)| \geq 1$. The $L(d,1)$-number of G, $\lambda_d(G)$, is defined as the minimum m such that there is an $L(d,1)$-labeling f of G with $f(V) \subseteq \{0,1,2,\ldots,m\}$. Motivated by the channel assignment problem introduced by Hale [9], the $L(2,1)$-labeling and the $L(1,1)$-labeling (as $d=2$ and 1, respectively) have been studied extensively in the past decade. This article extends the study to all positive integers d. We prove that $\lambda_d(G) \leq \Delta^2+(d-1)\Delta$ for any graph G with maximum degree Δ. Different lower and upper bounds of $\lambda_d(G)$ for some families of graphs including trees and chordal graphs are presented. In particular, we show that the lower and the upper bounds for trees are both attainable, and the upper bound for chordal graphs can be improved for several subclasses of chordal graphs.

Keywords. Vertex-coloring, distance two labeling, channel assignment problem, $L(2,1)$-labeling.

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1 Introduction

The $L(2,1)$-labeling, proposed by Griggs and Roberts [13], arose from a variation of the channel assignment problem introduced by Hale [9]. Suppose a number of transmitters or stations are given. We ought to assign a channel to each of the given transmitters or stations such that the interference is avoided. In order to reduce the interference, any two "close" transmitters must receive different channels, and any two "very close" transmitters (between which stronger interference may occur) must receive channels by at least two apart. One can construct an interference graph for this problem so that each transmitter or station is represented by a vertex on $\Re^2$, and there is an edge between two "very close" transmitters or stations. Two transmitters or stations are defined "close" if the corresponding vertices are of distance two, that is, a shortest path between those two vertices has two edges.

Thus, for a given graph G, an $L(2,1)$-*labeling* of G is defined as a function $f : V(G) \to \{0, 1, 2, 3, \ldots\}$ such that $|f(u)-f(v)| \geq 2$ if $uv \in E(G)$; and $|f(u)-f(v)| \geq 1$ if $\mathrm{d}_G(u,v) = 2$, where $\mathrm{d}_G(u,v)$, the *distance* of u and v, is the length (number of edges) of a shortest path between u and v.

For a more general setting, Griggs and Yeh [8] proposed the study of labelings f such that $|f(x) - f(y)| \geq m_i$ if $d_G(x,y) = i$ for $1 \leq i \leq N$, where N is a positive integer and $m_1 \geq m_2 \geq \ldots \geq m_N > 0$ are given numbers. If $N = m_1 = 1$, it is the same as ordinary vertex coloring. If $N = 2, m_1 = d$ and $m_2 = k$, then it is the $L(d,k)$-labeling. That is, for a given G and positive integers $d \geq k$, an $L(d,k)$-*labeling* of G is a non-negative integral function f on $V(G)$ such that the following are satisfied:

$$|f(u) - f(v)| \geq \begin{cases} d, & \text{if } \mathrm{d}_G(u,v) = 1; \\ k, & \text{if } \mathrm{d}_G(u,v) = 2. \end{cases}$$

Given G and positive integers $d \geq k$, an m-$L(d,k)$-*labeling* is an $L(d,k)$-labeling so that no label used is greater than m. The $L(d,k)$-*number* of G, denoted by $\lambda_{d,k}(G)$, is the smallest number m such that G has an m-$L(d,k)$-labeling. An m-$L(d,k)$-labeling

f is *optimal* if $m = \lambda_{d,k}(G)$, and in this case f is also called a $\lambda_{d,k}$-labeling.

Georges and Mauro [5] studied the $L(d,k)$-labeling for general values of d and k, $d \geq k$. In [5], among other results, the authors completely determined the exact values (for all d and k with $d \geq k$) of $\lambda_{d,k}(G)$ for special families of graphs including paths and cycles; partially determined the values of $\lambda_{d,k}(G)$ for products of paths; and gave the value of $\lambda_{d,k}(T)$ for some trees T of maximum degree $\Delta \leq d/k$.

In this article, we focus on $L(d,1)$-labelings and denote $\lambda_{d,1}(G)$ by $\lambda_d(G)$, unless indicated. (Note that $\lambda_2(G)$ is also denoted by $\lambda(G)$ in the literature.) In addition, we study a variation of $L(d,1)$-labeling, namely, $L'(d,1)$-labeling which is a *one-to-one* $L(d,1)$-labeling. The $L'(d,1)$-number of a graph G denoted by $\lambda'_d(G)$ is the minimum m such that there is an m-$L'(d,1)$-labeling for G. The exact values of $\lambda'_d(G)$ for paths, cycles and complete multipartite graphs were given by Georges and Mauro [5].

In Section 2, we show some general properties about $\lambda_d(G)$ and $\lambda'_d(G)$ and their relations with other parameters of G such as $\chi(G)$, the chromatic number of G (the minimum number of colors of a *proper-coloring* of $V(G)$, i.e., adjacent vertices receive different colors), and $\Delta(G)$, the maximum degree of a vertex in G.

The $L(2,1)$-labeling and the $L(1,1)$-labeling of graphs have been extensively studied in the past decade (cf. [1, 6, 8, 10, 11, 12, 13, 14, 15]). For any graph G with maximum degree Δ, Griggs and Yeh [8] proved that $\lambda_2(G) \leq \Delta^2 + 2\Delta$ and conjectured that $\lambda_2(G) \leq \Delta^2$; Chang and Kuo [1] improved the bound to $\lambda_2(G) \leq \Delta^2 + \Delta$. For $L(d,1)$-labeling, we prove, in Section 2, that for any graph G and $d \in Z^+$, $\lambda_d(G) \leq \Delta^2 + (d-1)\Delta$.

Section 3 focuses on chordal graphs and several subclasses of chordal graphs including r-paths, t-trees, trees and strongly chordal graphs. The $L(2,1)$-labeling for chordal graphs was investigated by Sakai [14], in which, it was proved that $\lambda_2(G) \leq (\Delta+3)^2/4$ for any chordal graph with maximum degree Δ. We prove that if G is a chordal graph with maximum degree Δ, then $\lambda_d(G) \leq (2d + \Delta - 1)^2/4$. Better upper bounds of

$\lambda_d(G)$ for several subclasses of chordal graphs are presented. Let T denote a tree with maximum degree $\Delta \geq 1$, it was proved in [8] that $\lambda_2(T)$ is either $\Delta + 1$ or $\Delta + 2$. Later on, Chang and Kuo [1] showed a polynomial algorithm determining the exact value of $\lambda_2(T)$ for any tree T. We will show that if T is a tree, then $\Delta + d - 1 \leq \lambda_d(T) \leq \min\{\Delta + 2d - 2, 2\Delta + d - 2\}$ for any $d \in Z^+$, and the lower and the upper bounds are both attainable. In addition, it is of polynomial time to determine the exact value of $\lambda_d(T)$ by using a modified algorithm shown in [1] in determining $\lambda_2(T)$.

2 General properties

In this section, we present basic properties about the $L(d, 1)$-number $\lambda_d(G)$ and the $L'(d, 1)$-number $\lambda'_d(G)$ and their relations with $\chi(G)$ and $\Delta(G)$ for any graph G. We give a general upper bound of $\lambda_d(G)$ in terms of $\Delta(G)$, and show that the asymptotic ratio $\lambda_d(G)/\lambda_{d+1}(G)$ approaches to 1 as $d \to \infty$.

For any fixed positive integer k, the *k-th power* of a graph G is the graph G^k with

$$V(G^k) = V(G) \text{ and } E(G^k) = \{xy : 1 \leq \mathrm{d}_G(x, y) \leq k\}.$$

It is an easy exercise to verify the following: (cf. [5, 12])

$$\lambda_{1,1}(G) = \chi(G^2) - 1 \text{ and } \chi(G) - 1 \leq \lambda_d(G) \leq d(\chi(G^2) - 1). \qquad (*)$$

Lemma 2.1 *If G is a graph with n vertices, then $\lambda'_1(G) = n - 1$; and $\lambda'_d(G) \geq n - 1$ as $d \geq 2$.*

Let P_n denote a path on n vertices, then the *r-path* of order n, P_n^r, is the r-th power of P_n. A *Hamiltonian r-path* is an r-path covering each vertex of G exactly once. Using the Hamiltonian r-path, the next result characterizes the sharpness of the inequality in Lemma 2.1.

Theorem 2.2 *Suppose G is a graph with n vertices. Then $\lambda'_d(G) = n-1$ if and only if G^c, the complement of G, has a Hamiltonian $(d-1)$-path.*

Proof. Suppose $\lambda'_d(G) = n-1$, then there exists an $L'(d,1)$-labeling such that $f(v_i) = i$ for $0 \le i \le n-1$, where $V(G) = \{v_0, v_1, ..., v_{n-1}\}$. If $1 \le |i-j| \le d-1$, then $1 \le |f(v_i) - f(v_j)| \le d-1$, so $v_iv_j \in E(G^c)$. Therefore, $\{v_0, v_1, ..., v_{n-1}\}$ forms a Hamiltonian $(d-1)$-path.

Suppose G^c has a Hamiltonian $(d-1)$-path: $v_0, v_1, ..., v_{n-1}$. Then the function $f : V(G) \to \{0, 1, ..., n-1\}$ defined by $f(v_i) = i$ gives an $L'(d,1)$-labeling of G, so $\lambda'_d(G) \le n-1$. By Lemma 2.1, $\lambda'_d(G) = n-1$. □

For any fixed positive integer k, a *k-stable set* of a graph G is a subset S of $V(G)$ such that every two distinct vertices in S are of distance greater than k. Note that 1-stability is the usual stability.

Next, we show a general upper bound of $\lambda_d(G)$ in terms of the maximum degree Δ of G for any $d \in Z^+$. The special case as $d = 2$ was proved in [1]. For completeness, we include the proof here which is analogous to the one in [1].

Theorem 2.3 *If G is a graph with maximum degree Δ, then $\lambda_d(G) \le \Delta^2 + (d-1)\Delta$ for $d \ge 2$.*

Proof. Consider the following labeling scheme on $V(G)$. Initially, all vertices are unlabeled and let $S_{-d+1} = S_{-d+2} = \ldots = S_{-1} = \emptyset$. For $i = 0, 1, 2, \cdots$, if $S_{i-d+1}, S_{i-d+2}, \ldots, S_{i-1}$ are determined but not all vertices in G are labeled, let

$$F_i = \{x \in V(G) : x \text{ is unlabeled and } d(x,y) \ge 2 \text{ for all } y \in \cup_{j=i-d+1}^{i-1} S_j\}.$$

Choose a *maximal* 2-stable subset S_i of F_i, i.e., S_i is a 2-stable subset of F_i but S_i is not a proper subset of any 2-stable subset of F_i. In the case where $F_i = \emptyset$, we have $S_i = \emptyset$. Label all vertices in S_i by i, and continue this process until all vertices are labeled. Assume k is the maximum label used, and choose a vertex x whose label is k. Let

$I_1 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) = 1 \text{ for some } y \in S_i\}$,

$I_2 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \leq 2 \text{ for some } y \in S_i\}$,

$I_3 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \geq 3 \text{ for all } y \in S_i\}$.

It is clear that $|I_2|+|I_3| = k$. Since the total number of vertices y with $1 \leq d(x,y) \leq 2$ is at most $\deg(x) + \sum\{\deg(y) - 1 : (y,x) \in E(G)\} \leq \Delta + \Delta(\Delta - 1) = \Delta^2$, we have $|I_2| \leq \Delta^2$. Also, there exist only $\deg(x) \leq \Delta$ vertices adjacent to x, so $|I_1| \leq \Delta$.

If $i \in I_3$, then $x \notin F_i$, for otherwise $S_i \cup \{x\}$ is a 2-stable subset of F_i, contradicting with the choice of S_i. Hence, $d(x,y) = 1$ for some vertex y in $\bigcup_{j=i-d+1}^{i-1} S_j$, i.e., there exists j with $i-d+1 \leq j \leq i-1$ such that $j \in I_1$. Define a function from I_3 to I_1 by mapping i to such a j with $i-d+1 \leq j \leq i-1$. Then each j in I_1 is the image of at most $d-1$ elements in I_3. This implies $|I_3| \leq (d-1)|I_1|$. So, we have

$$\lambda_d(G) \leq k = |I_2| + |I_3| \leq |I_2| + (d-1)|I_1| \leq \Delta^2 + (d-1)\Delta. \qquad \square$$

Theorem 2.4 *If G is a graph with at least one edge, then* $\lim\limits_{d \to \infty} \frac{\lambda_{d+1}(G)}{\lambda_d(G)} = 1$.

Proof. Suppose f is a λ_d-labeling of G. Consider the labeling g of G defined by $g(x) = \lfloor \frac{(d+1)f(x)}{d} \rfloor$, $x \in V(G)$. Suppose $xy \in E(G)$, then $|f(x) - f(y)| \geq d$. Without loss of generality, assume $f(x) - f(y) \geq d$. Then we have $\frac{(d+1)f(x)}{d} - \frac{(d+1)f(y)}{d} \geq d+1$, so $|g(x) - g(y)| \geq d+1$.

If x and y are of distance two in G, then $|f(x) - f(y)| \geq 1$ which implies $|g(x) - g(y)| \geq 1$. Hence, g is an $L(d+1,1)$-labeling of G, so $\lambda_{d+1}(G) \leq \lfloor \frac{(d+1)\lambda_d(G)}{d} \rfloor$. Therefore, we have $1 \leq \frac{\lambda_{d+1}(G)}{\lambda_d(G)} \leq \frac{d+1}{d}$, so $\lim\limits_{d \to \infty} \frac{\lambda_{d+1}(G)}{\lambda_d(G)} = 1$. $\square$

The proof above was done by showing an appropriate labeling. A different proof is due to one referee: By the result of Georges and Mauro [5], "For any G and any positive integer c, $c\lambda_{d,k}(G) = \lambda_{cd,ck}(G)$," we obtain

$$d\lambda_{d+1,1}(G) = \lambda_{d^2+d,d}(G) \leq \lambda_{d^2+d,d+1}(G) = (d+1)\lambda_{d,1}(G).$$

It then follows that $1 \le \frac{\lambda_{d+1}(G)}{\lambda_d(G)} \le \frac{d+1}{d}$ and $\lim_{d\to\infty} \frac{\lambda_{d+1}(G)}{\lambda_d(G)} = 1$.

3 Chordal graphs

A graph is *chordal* (or *triangulated*) if every cycle of length greater than three has a *chord*, which is an edge joining two non-consecutive vertices of the cycle. Chordal graphs have been extensively studied as a subclass of perfect graphs (see [7]).

This section studies $\lambda_d(G)$ for several subclasses of chordal graphs. We start with r-paths and t-trees. The study on t-trees also leads to results for general trees (Theorem 3.4) and an upper bound of $\lambda_d(G)$ for any chordal graph G (Theorem 3.5). In addition, we show that the upper bound can be improved for some subclasses of chordal graphs.

Theorem 3.1 *If $r \ge 2$, then $\lambda_1(P_n^r) = \min\{n-1, 2r\}$, and for $d \ge 2$,*

$$\lambda_d(P_n^r) = \begin{cases} (n-1)d, & \text{if } n \le r+1; \\ rd+1, & \text{if } r+2 \le n \le 2r+2; \\ rd+2, & \text{if } n \ge 2r+3. \end{cases}$$

Proof. If $d = 1$, by (*), $\lambda_1(P_n^r) = \chi((P_n^r)^2) - 1 = \chi(P_n^{2r}) - 1 = \min\{n-1, 2r\}$.

Suppose $d \ge 2$. If $n \le r+1$, then $\lambda_d(P_n^r) = (n-1)d$, since $P_n^r = K_n$. If $r+2 \le n \le 2r+2$, we have the following $L(d,1)$-labeling for P_{2r+2}^r:

$$\underbrace{1,\ \ d+1,\ \ 2d+1,\ \ \ldots\ldots\ ,\ \ rd+1}_{(r+1)\text{-terms}},\ \ \underbrace{0,\ \ d,\ \ 2d,\ \ \ldots\ldots\ ,\ \ rd}_{(r+1)\text{-terms}}.$$

Hence, $\lambda_d(P_{2r+2}^r) \le rd+1$. In P_n^r, every $r+1$ consecutive vertices form a clique, so $\lambda_d(P_n^r) \ge rd$. Suppose $\lambda_d(P_{r+2}^r) = rd$. Let f be a λ_d-labeling of P_{r+2}^r and let $V(P_{r+2}^r) = \{v_1, v_2, \ldots, v_{r+2}\}$. Then any $r+1$ consecutive vertices must use the labels $\{0, d, 2d, \ldots, rd\}$. This implies $f(v_1) = f(v_{r+2})$, a contradiction, since v_1 and v_{r+2} are of distance two in P_{r+2}^r. Therefore, we obtain

$$rd+1 \le \lambda_d(P_{r+2}^r) \le \lambda_d(P_n^r) \le \lambda_d(P_{2r+2}^r) \le rd+1,$$

so $\lambda_d(P_n^r) = rd + 1$ for $r + 2 \le n \le 2r + 2$.

If $n \ge 2r + 3$, label the first $2r + 2$ vertices by

$$\underbrace{0,\ \ d+2,\ \ 2d+2,\ \ \ldots\ldots\ \ ,\ \ rd+2}_{(r+1)\text{-terms}},\ \ \underbrace{1,\ \ d+1,\ \ 2d+1,\ \ \ldots\ldots\ \ ,\ \ rd+1}_{(r+1)\text{-terms}};$$

and then repeat the pattern of the last $(r + 1)$ terms to the remaining vertices. This gives an $L(d, 1)$-labeling, so $\lambda_d(P_n^r) \le rd + 2$. By the discussion in the previous paragraph, $\lambda_d(P_n^r) \ge \lambda_d(P_{2r+2}^r) = rd + 1$. It suffices to show $\lambda_d(P_{2r+3}^r) > rd + 1$. Suppose to the contrary, $\lambda_d(P_{2r+3}^r) = rd + 1$ and let f be a λ_d-labeling of P_{2r+3}^r. There are $r + 3$ cliques of size $r + 1$ in P_{2r+3}^r: $D_i = \{v_i, v_{i+1}, \ldots, v_{i+r}\}$, $1 \le i \le r + 3$. For each i, $f(D_i)$ must be one of the following $r + 2$ sets:

$$\{0, d, \ldots, jd, (j + 1)d + 1, (j + 2)d + 1, \ldots, rd + 1\}, -1 \le j \le r.$$

By the pigeonhole principle, $f(D_i) = f(D_j)$ for some $1 \le i < j \le r+3$. However, the index of v_{i+r} differs from the index of any vertex in D_j by at most $r+2 \le 2r$, so v_{i+r} is within distance two from any vertex of D_j. Hence, $f(v_{i+r}) \notin f(D_j)$ contradicting with $f(D_i) = f(D_j)$. □

We now turn to another subclass of chordal graphs [7], namely, t-trees. Given a positive integer t, t-trees are defined recursively by the following two rules:

(T1) K_t is a t-tree; and

(T2) if H is a t-tree, then the graph obtained from H by adding a new vertex joining to a t-clique (i.e. K_t) of H is a t-tree.

For any $a, b \in Z$, let $[a, b]$ denote the set of integers $\{a, a + 1, a + 2, \cdots, b\}$.

Theorem 3.2 *If G is a t-tree with maximum degree Δ, then $\lambda_d(G) \le (2d-1+\Delta-t)t$.*

Proof. We prove the theorem by induction on the number of vertices of G. The theorem is trivial as $G = K_t$.

Suppose G is obtained from a t-tree H by adding a new vertex v joining to a t-clique in H. By the induction hypothesis, H has an $L(d,1)$-labeling using labels from $[0,(2d-1+\Delta-t)t]$. It suffices to extend this labeling to an $L(d,1)$-labeling for G by assigning v an appropriate label from $[0,(2d-1+\Delta-t)t]$.

Each of the t neighbors of v eliminates at most $(2d-1)$ possible choices for the label of v. In addition, there are at most $(\Delta-t)t$ vertices with distance two away from v, and each of them eliminates at most one choice for the label of v. Therefore, at most $(2d-1+\Delta-t)t$ labels, totally, can not be used for v. Hence there is still at least one valid label in $[0,(2d-1+\Delta-t)t]$ for v. This completes the proof. $\square$

Note that any tree T is an 1-tree. Hence by Theorem 3.2, $\lambda_d(T) \le \Delta+2d-2$ for any tree T with maximum degree Δ. However, with the next theorem, we show that this upper bound for $\lambda_d(T)$ could be improved to $2\Delta+d-2$ as $\Delta \le d$. And, in any case, the upper bound is attainable.

A vertex v in G is called *major* if $\deg(v)=\Delta(G)$, where $\deg(v)$ is the degree of v in G. Suppose G is a graph of maximum degree Δ and let x be a major vertex, i.e., x is the center of a star $K_{1,\Delta}$ in G. If f is an $L(d,1)$-labeling for G, then $\mathrm{N}(x)$, neighbors of x, must receive distinct labels, since they are of distance at most two, and $|f(x)-f(y)| \ge d$ for any $y \in \mathrm{N}(x)$. Hence, $\lambda_d(G) \ge \Delta+d-1$. Indeed, by definition, one can prove the following:

Lemma 3.3 *If G is a graph of maximum degree $\Delta \ge 1$, then $\lambda_d(G) \ge \Delta+d-1$. Moreover, if $\lambda_d(G)=\Delta+d-1$ and $d \ge 2$, then $f(x)=0$ or $\Delta+d-1$ for any λ_d-labeling f of G and any major vertex x; consequently, it is impossible to have a set of three major vertices such that any two of them are of distance at most two apart.*

Theorem 3.4 *For any tree T of maximum degree Δ,*

$$\Delta+d-1 \le \lambda_d(T) \le \min\{\Delta+2d-2, 2\Delta+d-2\}.$$

Moreover, the lower and the upper bounds for $\lambda_d(T)$ are both attainable.

Proof. For the lower and the upper bounds of $\lambda_d(T)$, by Lemma 3.3 and Theorem 3.2, it suffices to show that $\lambda_d(T) \leq 2\Delta + d - 2$ for $\Delta \leq d$. Since T is a tree, we regard T as a bipartite graph with partition sets A and B. Because Δ is the maximum degree and $\Delta \leq d$, we can get an $L(d,1)$-labeling for T by using $[0, \Delta - 1]$ to label the vertices in A, and $[\Delta + d - 1, 2\Delta + d - 2]$ to label B. Thus, $\lambda_d(T) \leq 2\Delta + d - 2$ for $\Delta \leq d$

The lower bound $\Delta + d - 1$ is tight for $K_{1,\Delta}$, since $K_{1,\Delta}$ is a tree and $\lambda_d(K_{1,\Delta}) = \Delta + d - 1$.

To show the upper bound is attainable, we consider the tree T_Δ with

$$V(T_\Delta) = \{v_1^0\} \cup \{v_j^i : 1 \leq i, j \leq \Delta\} \text{ and}$$

$$E(T_\Delta) = \{v_1^0 v_1^i,\ v_1^i v_j^i : 1 \leq i \leq \Delta, 2 \leq j \leq \Delta\}.$$

Figure 1 shows an example of T_4.

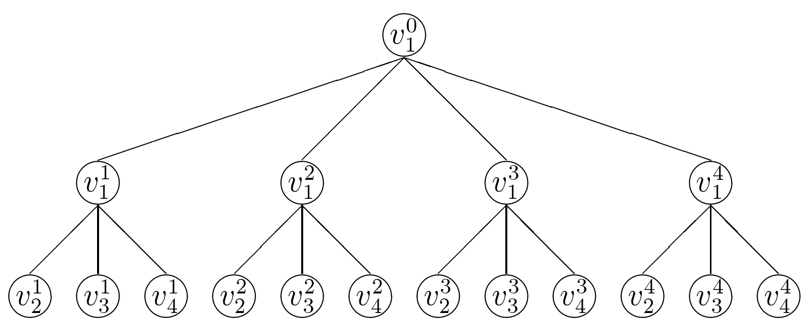

Figure 1. The tree T_4.

Let $m = \min\{d, \Delta\}$, it is enough to prove $\lambda_d(T_\Delta) = \Delta + d + m - 2$. Suppose, to the contrary, $\lambda_d(T_\Delta) = k \leq \Delta + d + m - 3$. Let f be a k-$L(d,1)$-labeling of T_Δ and let $M = \{v_1^i \in V : 0 \leq i \leq \Delta\}$. Note that every vertex v in M is major, i.e., $\deg(v) = \Delta$.

Claim. For any $v \in M$, $f(v) \in [0, m-2] \cup [\Delta + d - 1, \Delta + d + m - 3]$.

Proof of Claim. Suppose $m - 1 \leq f(v) \leq \Delta + d - 2$ for some $v \in M$. Let $a_1, a_2, \ldots, a_\Delta$ be the neighbors of v, where $f(a_1) < f(a_2) < f(a_3) < \ldots < f(a_\Delta)$. If

$f(a_{j-1}) < f(v) < f(a_j)$ for some $2 \le j \le \Delta$, then $f(a_\Delta) \ge f(a_1) + 2d + \Delta - 2 \ge 2d + \Delta - 2 \ge \Delta + d + m - 2$, a contradiction. Therefore, either $f(v) < f(a_1)$ or $f(v) > f(a_\Delta)$. The former case implies $f(a_\Delta) \ge f(v) + \Delta + d - 1 \ge \Delta + d + m - 2$, a contradiction; and the latter case implies $f(v) \ge f(a_1) + \Delta + d - 1 \ge \Delta + d - 1$, a contradiction. This proves the Claim.

By the Claim, $f(v) \in A_0 \cup A_1$ for all $v \in M$, where $A_0 = [0, m-2]$ and $A_1 = [\Delta + d - 1, \Delta + d + m - 3]$. In particular, $f(v_1^0) \in A_i$ where $i = 0$ or 1. Then $f(v_1^1)$, $f(v_1^2)$, ..., $f(v_1^\Delta)$ are all distinct and are all in A_{1-i}. This is a contradiction, since each A_i contains only $m - 1 \le \Delta - 1$ numbers. □

The proof of the theorem above implies that a tree T of maximum degree Δ has $\lambda_d(T) = \min\{\Delta + 2d - 2, 2\Delta + d - 2\}$, if there is a major vertex v in T such that all neighbors of v are major. Georges and Mauro [5] proved that if G is a graph of maximum degree Δ, and G has a major vertex such that all the neighbors of G are major, than $\lambda_{d,k}(G) \ge d + k(2\Delta - 2)$ if $d/k \ge \Delta$; $\lambda_{d,k}(G) \ge 2d + k(\Delta - 2)$ if $d/k \le \Delta$. Thus the second part of Theorem 3.4 can also be obtained by combining this result (letting $k = 1$) with the first past of Theorem 3.4. In addition, in [5], the authors showed that if T is a tree of maximum degree Δ and T has a major vertex v such that all neighbors of v are major, then $\lambda_{d,k}(T) = d + k(2\Delta - 2)$, if $\Delta \le d/k$. When $k = 1$, this result can be regarded as one case of the second part of Theorem 3.4.

It is known [8] that $\Delta + 1 \le \lambda_2(T) \le \Delta + 2$, and there is a polynomial algorithm [1] to determine the exact value of $\lambda_2(T)$ for any tree T. We note that the algorithm can be modified to determine $\lambda_d(T)$ simply by replacing the condition $|a - b| \ge 2$ in Theorem 6.1 of [1] with $|a - b| \ge d$. The modified algorithm also runs in a polynomial time.

It is known [7] that the vertex set of a chordal graph G has an ordering $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that for any i, $\mathrm{N}(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ forms a clique, C_i. Let $t = \max_{1 \le i \le n} |C_i|$. Applying the same argument in the proof of Theorem 3.2, we have

$\lambda_d(G) \le (2d-1+\Delta-t)t$, for any chordal graph G. Furthermore, the maximum value of $(2d-1+\Delta-t)t$ occurs as $t=\frac{2d+\Delta-1}{2}$. Therefore, the following result is obtained.

Theorem 3.5 *If G is a chordal graph with maximum degree Δ, then $\lambda_d(G) \le \frac{(2d+\Delta-1)^2}{4}$.*

In the rest of the section, we show the upper bound of $\lambda_d(G)$ for chordal graphs in Theorem 3.5 indeed can be improved from a quadratic to a linear function of Δ for several subclasses of chordal graphs.

An *n-sun* is a chordal graph with a Hamiltonian cycle $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1)$ in which each x_i has degree two. An *SF-chordal* (respectively, *OSF-chordal, 3SF-chordal*) graph is a chordal graph containing no n-sun with $n \ge 3$ (respectively, odd $n \ge 3$, $n = 3$) as an induced subgraph, where SF (respectively, OSF, 3SF) stands for sun-free (respectively, odd-sun-free, 3-sun-free). SF-chordal graphs are also called *strongly chordal* graphs by Farber [4]. Strongly chordal graphs include directed path graphs, interval graphs, unit interval graphs, block graphs, and trees.

A vertex x is *simple* if $N[y] \subseteq N[z]$ or $N[z] \subseteq N[y]$ for any two vertices $y, z \in N[x]$, where $N[v] := \{w : w = v \text{ or } wv \in E\}$. Consequently, if x is simple, then $N[x]$ is a clique, and x has a *maximum neighbor* $y \in N[x]$, i.e., $N[z] \subseteq N[y]$ for any $z \in N[x]$. Farber [4] proved that G is a strongly chordal graph if and only if every vertex-induced subgraph of G has a simple vertex.

For an OSF-chordal graph G, it is proved [3] that $\chi(G^2) = \omega(G^2)$, where ω is the size of a maximum clique in G. If G is OSF-chordal, then G is 3SF-chordal, by Theorem 3.8 of [2], $\omega(G^2) = \Delta + 1$. Therefore, combining with (*), we have the following result:

Theorem 3.6 *If G is an OSF-chordal graph with maximum degree Δ, then $\lambda_d(G) \le d\Delta$.*

Theorem 3.7 *If G is a strongly chordal graph with maximum degree Δ, then $\lambda_d(G) \leq \Delta + (2d-2)(\chi(G)-1)$.*

Proof. We prove the theorem by induction on $|V(G)|$. The theorem is obvious when $|V(G)| = 1$. Suppose $|V(G)| \geq 2$. Choose a simple vertex v of G. Since $G - v$ is also strongly chordal, by the induction hypothesis,

$$\lambda_d(G-v) \leq \Delta(G-v) + (2d-2)(\chi(G-v)-1) \leq \Delta + (2d-2)(\chi(G)-1).$$

Let f be a $\lambda_d(G-v)$-$L(d,1)$-labeling of $G-v$ and let m be the maximum neighbor of v, then every vertex of distance two from v is adjacent to m. Hence, there are $\deg(m) - \deg(v)$ vertices that are of distance two from v. Therefore, the number used by f to be avoided for v is at most

$$(2d-1)\deg(v) + \deg(m) - \deg(v) \leq \Delta + (2d-2)(\omega(G)-1) = \Delta + (2d-2)(\chi(G)-1).$$

Hence, there is at least one number in $[0, \Delta + (2d-2)(\chi(G)-1)]$ that can be assigned to v in order to extend f into a $(\Delta + (2d-2)(\chi(G)-1))$-$L(d,1)$-labeling for G. $\Box$

Note that although a strongly chordal graph is OSF-chordal, the upper bounds in Theorems 3.6 and 3.7 are incomparable.

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