California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Fall 2003

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Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}$ denotes the set of real numbers.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

$\bar{z}$ denotes the complex conjugate of the complex number z.

$|z|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\, dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Fall 2003 # 1. For each of the following, determine if it is a vector space over $\mathbb{R}$. Give reasons for your answers.

a. The set of all integrable real valued functions f on $[0,1]$ with $\int_0^1 f(t)\,dt = 0$.
b. The set of all polynomials with real coefficients and even degree.
c. The set of all differentiable real valued functions on $\mathbb{R}$ with $f(0) + f'(0) = 0$ and $f(1) - f'(1) = 0$.
d. The set of all 2×2 matrices with real entries and determinant equal to 0.

Fall 2003 # 2. For continuous complex valued functions f and g on the interval $[0,1]$, put

$$[f,g] = \int_0^1 f(t)\overline{g(t)}e^t\,dt.$$

a. Show that this defines an inner product on the space $C([0,1],\mathbb{C})$ of continuous complex valued functions on $[0,1]$.
(You may use your knowledge of the known inner product $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt$ on that space.)
b. Find polynomials $q_0(x)$ and $q_1(x)$ which are orthonormal with respect to the inner product $[\cdot,\cdot]$ and which span the space of polynomials of degree no more than 1.

(You may use: $\int_0^1 e^t\,dt = e - 1$; $\int_0^1 te^t\,dt = 1$; and $\int_0^1 t^2e^t\,dt = e - 2$.)

Fall 2003 # 3. Let a be a positive real constant. For x in $[-\pi,\pi]$, put $f(x) = e^{ax}$.

a Compute either the exponential or the trigonometric form of the Fourier series for f on $[-\pi,\pi]$. (Your choice which).
b Use the result of part **a.** to show that

$$\sum_{k=1}^{\infty} \frac{1}{a^2+k^2} = \frac{\pi}{2a}\,\frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}} - \frac{1}{2a^2}$$

Fall 2003 # 4. Suppose $\{e_1, e_2, e_3, \dots\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ and $\lambda_1, \lambda_2, \lambda_3, \dots$ is a bounded sequence of numbers with $\Lambda = \sup\{|\lambda_k| : k = 1,2,3,\dots\}$. For f in $\mathcal{H}$, let

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_{k+1} \rangle e_k.$$

a. Show that the series for T converges to an element of $\mathcal{H}$.
b. Show that T is a linear operator on $\mathcal{H}$.
c. Show that T is bounded as a linear operator on $\mathcal{H}$.
d. Show that the operator norm $\|T\|$ is equal to Λ.

(If you have trouble with part **a**, go ahead and do the rest assuming that the limit exists.)

Fall 2003 # 5. For each continuous numerical valued function f on $[0,1]$ define Tf by

$$(Tf)(x) = \sin x + \lambda \int_0^x (x - t^2) f(t)\, dt.$$

a. Find a range of values of λ for which T is a contraction with respect to the supremum norm on $C([0,1])$. Justify your answer.
b. Find a range of values of λ for which T is a contraction with respect to the L^2-norm on $C([0,1])$. Justify your answer.
c. Show that solutions f to the equation $f(x) = (Tf)(x)$ satisfy the ordinary differential equation

$$f''(x) + \lambda x(x-1) f'(x) + 2\lambda(x-1) f(x) = -\sin x$$

Fall 2003 # 6. For f in $L^2([0,1])$ define Kf by $(Kf)(x) = \displaystyle\int_0^1 (1 + 5x^2t^2) f(t)\, dt.$

a. Find all nonzero eigenvalues of K and the corresponding eigenfunctions (eigenvectors).

b. Find a function $R(x,t,\lambda)$ such that solutions to the equation

$$(*) \qquad f(x) = g(x) + \lambda \int_0^1 (1 + 5x^2t^2) f(t)\, dt$$

are given by

$$f(x) = g(x) + \lambda \int_0^1 R(x,t,\lambda) g(t)\, dt$$

c. Solve the equation (*) when $\lambda = 1$ and $g(x) = 1$ for all x.

Fall 2003 # 7. Consider the boundary value problem: For known function f on $[0,\pi]$, find a function y on $[0,\pi]$ such that

$$y''(x) + y(x) = f(x) \quad \text{for } 0 \le x \le \pi \qquad \text{with } y(0) = y'(\pi) = 0.$$

a. Find a function $G(x,t)$ such that solutions to this problem are given by

$$y(x) = \int_0^\pi G(x,t) f(t)\, dt.$$

b. Use the result of part **a** to solve for $y(x)$ when $f(x) = x$.

End of Exam