California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Fall 2004

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Do **five** of the following eight problems. Each problem is worth 20 points.

If you attempt more than 5, the best 5 will be counted.

Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}$ denotes the set of real numbers.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

$\bar{z}$ denotes the complex conjugate of the complex number z.

$|z|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\,dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a\cos b + \cos a \sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Fall 2004 # 1. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$.

Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$ and v a vector in $\mathcal{H}$.

We say the sequence $\{v_n\}_{n=1}^{\infty}$ converges strongly to v if $\lim\limits_{n\to\infty} \| v_n - v \| = 0$.

We say the sequence $\{v_n\}_{n=1}^{\infty}$ converges weakly to v if $\lim\limits_{n\to\infty} \langle v_n, w \rangle = \langle v, w \rangle$ for every w in $\mathcal{H}$

- **a.** Show that if the sequence $\{v_n\}_{n=1}^{\infty}$ converges strongly to v, then it also converges weakly to v.
- **b.** Show that if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $\mathcal{H}$, then $\{e_n\}_{n=1}^{\infty}$ converges weakly to the zero vector.
- **c.** Show that in an infinite dimensional Hilbert space weak convergence of a sequence to a limit v does not imply strong convergence of the sequence to v.

Fall 2004 # 2. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator from a Hilbert space $\mathcal{H}$ into itself.

Let $\text{range}(T) = \{Tx : x \in \mathcal{H}\}$ Let $\ker(T) = \{x \in \mathcal{H} : Tx = 0\}$

- **a.** (5 pts) Show that $\text{range}(T)$ and $\ker(T)$ are vector subspaces of $\mathcal{H}$.
- **b.** (5 pts) Show that $\ker(T)$ is a closed subset of $\mathcal{H}$.
- **c.** (3 pts) Give a definition of the orthogonal complement, $A^{\perp}$, of a subset A of $\mathcal{H}$.
- **d.** (7 pts) Show that $\ker(T) = (\text{range}(T^*))^{\perp}$

Fall 2004 # 3. For f in $L^2([-1,1])$, define Kf by $(Kf)(x) = \displaystyle\int_{-1}^{1} (5x + 7x^3t^3) f(t)\, dt$.

- **a.** Find any nonzero eigenvalues and the associated eigenvectors for the integral operator K.
- **b.** Find a function $R(x,t;\lambda)$ such that solutions f to the integral equation

$$f(x) = g(x) + \lambda \int_{-1}^{1} (5x + 7x^3t^3) f(t)\, dt$$

for a known function g are given by

$$f(x) = g(x) + \lambda \int_{-1}^{1} R(x,t;\lambda) g(t)\, dt.$$

Fall 2004 # 4. Let $\mathcal{P}^2$ be the space of all polynomials with real coefficients.

Use the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$ on $\mathcal{P}^2$.

- **a.** Find a basis for $\mathcal{P}^2$ which is orthonormal with respect to this inner product.
- **b.** Find constants a, b, and c which minimize the quantity $J = \int_0^1 \left| t^4 - a - bt - ct^2 \right|^2\, dt$

Fall 2004 # 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = -1$ for $-\pi < x < 0$, $f(x) = 1$ for $0 \le x \le \pi$, and extending so that f is 2π-periodic.

a. Compute the Fourier series for f. (Either the exponential or trigonometric form, your choice.)

b. Use the result of part **(a)** to show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

Fall 2004 # 6. Let $\mathcal{V}$ the space $C([-\pi, \pi])$ of all continuous complex valued functions on $[-\pi, \pi]$

For f and g in $\mathcal{V}$, let $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$.

For f in $\mathcal{V}$, let $\phi(f) = f(0)$. For integers k, let $e_k(t) = e^{ikt}$.

For positive integer n, let $\mathcal{M}_n$ be the space spanned by $\{e_k : -n \le k \le n\}$.

a. Show that ϕ is a linear functional on $\mathcal{V} = C([-\pi, \pi])$.

b. Show that ϕ is continuous when the uniform norm, $\| f \|_\infty = \sup\{|f(t)| : t \in [-\pi, \pi]\}$, is used on $C([-\pi, \pi])$.

c. Find trigonometric polynomials $q_n(t) = \sum_{j=-n}^{n} \lambda_j e_j(t)$ in $\mathcal{M}_n$ such that $\phi(f) = \langle f, q_n \rangle$ for all f in $\mathcal{M}_n$.

d. Show that when the L^2-norm associated with $\langle \cdot, \cdot \rangle$ is used on $C([-\pi, \pi])$, then the operator norm of the restriction of ϕ to $\mathcal{M}_n$ is $\sqrt{2n+1}$.

e. Show that ϕ is not continuous on $C([-\pi, \pi])$ when the norm of part **d** is used on $C([-\pi, \pi])$.

Fall 2004 # 7. Suppose p is a differentiable function on $[a, b]$ with p' continuous and $p(x) > 0$ for all x in $[a, b]$. Let L be the differential operator defined for twice differentiable functions f on $[a, b]$ by $(Lf)(x) = -\frac{d}{dx}[p(x)f'(x)]$

Let $\mathcal{V}$ be the space of all real valued twice differentiable functions f on $[a, b]$ with f'' continuous and $f(a) = f(b) = 0$. With the inner product $\langle f, g \rangle = \int_a^b f(t)g(t)\, dt$ on $\mathcal{C}([a, b], \mathbb{R})$, prove each of the following

a. $\langle Lf, g \rangle = \langle f, Lg \rangle$ for all f and g in $\mathcal{V}$.

(Suggestion: Compute each side separately and compare the results.)

b. If $\lambda \ne \mu$ are distinct real numbers and f and g are in $\mathcal{V}$ with $Lf = \lambda f$ and $Lg = \mu g$, then $\langle f, g \rangle = 0$.

(Note: Do not just quote a known fact about self-adjoint operators)

End of Exam