California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Fall 2005

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.
$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\,dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a\cos b + \cos a \sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Fall 2005 # 1. Let a be a nonzero real constant and let f be the function defined by putting

$$f(t) = \begin{cases} e^{at} \text{ for } -\pi < t < \pi \\ 0 \text{ for } t = \pm\pi \end{cases}$$

and extending to be 2π-periodic.

a. Compute the Fourier series for f. (Trigonometric or exponential, your choice, but the exponential is probably easier.)

b. Show that $\dfrac{1}{a^2} + 2\displaystyle\sum_{k=1}^{\infty} \frac{1}{a^2+k^2} = \frac{\pi}{a}\frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}}$

Fall 2005 # 2. For each of the following families of functions decide and prove whether they are or are not linearly independent as functions on the interval $[-\pi, \pi]$.

a. $\mathcal{A} = \{f_1, f_2, f_3\}$ where $f_1(x) = 1$, $f_2(x) = \cos x$, and $f_3(x) = \sin x$ for all x.

b. $\mathcal{B} = \{g_1, g_2, g_3\}$ where $g_1(x) = 1$, $g_2(x) = \cos^2 x$, and $g_3(x) = \sin^2 x$ for all x.

Fall 2005 # 3. Let $\mathcal{V}$ be the space of all continuous real valued functions on $[-\pi, \pi]$ with the norm $\| f \| = (1/\pi) \int_{-\pi}^{\pi} |f(t)|^2 \, dt$.

For each positive integer n, let $\mathcal{T}_n$ be the subspace of $\mathcal{V}$ consisting of all trigonometric polynomials with real coefficients and order no more than n.

$$\mathcal{T}_n = \{q(t) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt) : a_k \in \mathbb{R} \text{ and } b_k \in \mathbb{R} \quad \forall k\}$$

Let $\mathcal{T}$ be the subspace of $\mathcal{V}$ consisting of all trigonometric polynomials with real coefficients of all orders.

Let D be the differentiation operator, $Df = f'$.

a. Show that D maps $\mathcal{T}_n$ into $\mathcal{T}_n$ for each n and that D maps $\mathcal{T}$ into $\mathcal{T}$.

b. Show that D is bounded as a linear operator from $\mathcal{T}_n$ into $\mathcal{T}_n$ and find the operator norm of the restriction $D|_{\mathcal{T}_n}$.

c. Show that D is not bounded as a linear operator from $\mathcal{T}$ into $\mathcal{T}$.

Fall 2005 # 4. Suppose $\mathcal{M}$ is a subset of a Hilbert space $\mathcal{H}$ and that T is a bounded linear operator from $\mathcal{H}$ into a Hilbert space $\mathcal{K}$.

a. Show that the orthogonal complement, $\mathcal{M}^{\perp}$, of $\mathcal{M}$ is a vector subspace of $\mathcal{H}$.

b. Show that $\ker(T) = \{f \in \mathcal{H} : Tf = 0\}$ is a vector subspace of $\mathcal{H}$.

c. Show that $\text{range}(T) = \{Tf \in \mathcal{K} : f \in \mathcal{H}\}$ is a vector subspace of $\mathcal{K}$.

d. Show that T maps $(\ker(T))^{\perp}$ one to one onto $\text{range}(T)$

Fall 2005 # 5. Let (x, y) and (a, b) represent vectors in $\mathbb{R}^2$.

a. For each of the following decide whether the formula given for $\|(x, y)\|$ defines a norm on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.

(i) $\|(x, y)\| = 2|x|$

(ii) $\|(x, y)\| = x^2 + y^2$

b. For each of the following decide whether the formula given for $\langle (a, b), (x, y) \rangle$ defines an inner product on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.

(i) $\langle (a, b), (x, y) \rangle = 2ax + 3by$

(ii) $\langle (a, b), (x, y) \rangle = 2ax - 3by$

Fall 2005 # 6. For each continuous function f on the interval $[0, 1]$, let the function Kf on $[0, 1]$ be given by $(Kf)(x) = \int_0^1 (2 + 6xt) f(t)\, dt$.

a. Find all nonzero eigenvalues and the corresponding eigenfunctions for the operator K.

b. Find a solution f to the integral equation $f(x) = 1 + \int_0^1 (2 + 6xt) f(t)\, dt$.

Fall 2005 # 7. For f in the space $C([0, 1])$ of all continuous real valued functions on the interval $[0, 1]$ and numerical parameter λ, let Tf be the function on $[0, 1]$ defined by

$$(Tf)(x) = 1 + \lambda \int_0^x xtf(t)\, dt.$$

a. Find a range of values of the parameter λ for which the transformation T is a contraction on $C([0, 1])$ with respect the the norm $\| f \|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$.

b. With $f_0 = 1$ for all x, find the next two iterates f_1 and f_2 in the iterative process for solving the integral equation

$$f(x) = 1 + \lambda \int_0^x xtf(t)\, dt.$$

c. If f is a solution to the integral equation in part **b**, then f is also a solution to a differential initial value problem of the form

$$f''(x) + a(x)f'(x) + b(x)f(x) = \phi(x) \quad \text{with } f(0) = A \text{ and } f'(0) = B$$

Find $a(x)$, $b(x)$, $\phi(x)$, A, and B.

End of Exam