California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Fall 2006

Cooper, Gutarts*, Hoffman, Verona

Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.

Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{N}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|\, z\,|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{N})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |\, f(x)\,|^2 \; dx < \infty$

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$ $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$ $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Fall 2006 # 1. **a.** Find the Fourier series for the function $f(x) = x$ on the interval $[-\pi, \pi]$.

(You may use either the trigonometric or exponential form.)

b. Use the result of part **a** to show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Fall 2006 # 2. For each continuous function f on the interval $[0, 1]$, let Tf be defined by

$$(Tf)(x) = x^2 + \lambda \int_0^x (x^2 - t^2) f(t)\, dt.$$

a. Find a range of values of λ for which T is a contraction with respect to the supremum norm, $\| f \|_\infty = \sup\{| f(t) | : t \in [0, 1]\}$, on the space $C([0, 1])$ of continuous functions on $[0, 1]$.

b. Find a range of values of λ for which T is a contraction with respect to the L^2-norm, $\| f \|_2 = \left(\int_0^1 | f(t) |^2 \, dt\right)^{1/2}$, on the space $C([0, 1])$ of continuous functions on $[0, 1]$.

c. Explain, with an explicit statement of the formula to be iterated, how to use the contraction mapping principle to generate a sequence of approximation to a solution f for the equation

$$f(x) = x^2 + \lambda \int_0^x (x^2 - t^2) f(t)\, dt.$$

Starting with $f_0(x) = 1$, compute the next two approximations, f_1 and f_2.

Fall 2006 # 3. For each continuous function f on the interval $[0, 1]$, let Kf be defined by

$$(Kf)(x) = \int_0^1 x^3 t^2 f(t)\, dt.$$

a. Find a function $R(x, t; \lambda)$ such that solutions to the equation $f = g + \lambda K f$ are given by

$$f(x) = g(x) + \lambda \int_0^1 R(x, t; \lambda) g(t)\, dt$$

for each continuous function g on $[0, 1]$.

b. Find a function f on $[0, 1]$ such that

$$f(x) = x + \int_0^1 x^3 t^2 f(t)\, dt \qquad \text{for each } x \text{ in } [0, 1].$$

Fall 2006 # 4. For specified functions h on $[0, 1]$, consider the boundary value problem

$$\text{(BVP)} \qquad -\frac{d}{dx}\left[e^x f'(x)\right] = h(x) \text{ for } x \text{ in } [0, 1] \qquad \text{with } f(0) = 0 \text{ and } f'(1) = 0.$$

a. Find a function $G(x, t)$ such that solutions to (BVP) are given by

$$f(x) = \int_0^1 G(x, t) h(t)\, dt.$$

b. Solve the problem (BVP) with $h(x) = 1$ for all x in $[0, 1]$.

Fall 2006 # 5. Which of the following are vector spaces over $\mathbb{R}$?
For each, give a yes or no answer, and explain how you know that it is or is not.
(You may assume that that spaces $C(\mathbb{R};\mathbb{R})$ and $\mathbb{R}^d$ of all continuous real valued functions on $\mathbb{R}$ and of all ordered d-tuples of real numbers are vector spaces over $\mathbb{R}$ with the usual operations)

a. $\mathcal{A} = \{f \in C(\mathbb{R};\mathbb{R}) : f''$ exists and $f''(x)+3f'(x)+5f(x) = 0$ for all x in $\mathbb{R}\}$
b. $\mathcal{B} = \{f \in C(\mathbb{R};\mathbb{R}) : f''$ exists and $f''(x)+3f'(x)+5f(x) = \cos x$ for all x in $\mathbb{R}\}$
c. $\mathcal{C} =$ the set of all 2×2 matrices with entries in $\mathbb{R}$ with the operations of matrix addition and scalar multiplication
d. $\mathcal{D} =$ the set of all 2×2 matrices with entries in $\mathbb{R}$ and determinant 0 with the operations of matrix addition and scalar multiplication

---

Fall 2006 # 6. a. (4 points) What is the dimension of the space of all linear operators from $\mathbb{R}^2$ to $\mathbb{R}^2$. (Justify your answer.)

b. (8 points) Show that if T is a linear operator from $\mathbb{R}^2$ into $\mathbb{R}^2$, then there is a nonzero polynomial p with $p(T) = 0$.

(Do not just quote a theorem from linear algebra. You must prove something here. This is easier than that theorem anyway.)

(Suggestion: Think about the set of powers of T.)

c. (8 points) Is the conclusion of part **b** true for every continuous linear operator from an infinite dimension Hilbert space into itself? (Prove or give a counterexample.)

(Suggestion: Consider the shift operator from ℓ^2 into ℓ^2 given by $S((x_1, x_2, x_3, \dots)) = (0, x_1, x_2, x_3, \dots)$ and the first basis vector $e_1 = (1, 0, 0, 0, \dots)$)

Fall 2006 # 7. Suppose U and T are bounded linear operators on a Hilbert space $\mathcal{H}$ over the complex numbers such that $U^*U = I$ and $T^* = -T$. Show each of the following.

a. If μ is an eigenvalue for U, then $|\,\mu\,| = 1$.
b. If λ is an eigenvalue for T, then λ is purely imaginary in the sense that there is a real number b with $\lambda = ib$. That is, $\mathrm{Re}(\lambda) = 0$.

Suggestion: What can you do with $\langle\, Uv, Uv\,\rangle$ and $\langle\, Tv, v\,\rangle$.

End of Exam