California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Linear Analysis Fall 2017

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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

Please

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}$ denotes the set of real numbers.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

$\bar{z}$ denotes the complex conjugate of the complex number z.

$|\, z\,|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |\, f(x)\,|^2 \; dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a\cos b + \cos a \sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Fall 2017 # 1. For each of the following sets decide whether or not it is a vector space over $\mathbb{R}$. Give clear "yes" or "no" answers and justify them.

(You may assume that $\mathbb{R}^3$ is a vector space and that the set of all differentiable real valued functions on $\mathbb{R}$ is a vector space over $\mathbb{R}$.)

a. (5 pts) $\mathcal{A} = \{f : \mathbb{R} \to \mathbb{R} : f'(0) + f(5) = 0\}$.
b. (5 pts) $\mathcal{B} = \{f : \mathbb{R} \to \mathbb{R} : f'(0) + f(5) = 5\}$.
c. (5 pts) $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 : x + y = z + 3\}$.
d. (5 pts) $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 : x + y = z\}$.

Fall 2017 # 2. Let $C([-1, 1])$ be the space of all continuous real valued functions on the interval $[-1, 1]$ with the supremum norm $\| f \|_\infty = \sup\{| f(t) | : t \in [-1, 1]\}$. For f in $C([-1, 1])$ let $\phi(f) = f(1) - f(-1)$.

a. (6 pts) Show that $\phi : C([-1, 1]) \to \mathbb{R}$ is linear.
b. (7 pts) Show that ϕ is continuous with respect to the specified norm.
c. (7 pts) Find the functional (operator) norm of ϕ.

Fall 2017 # 3. Let P_1 be the space of all polynomials with real coefficients $ax + b$ of degree no more than 1 with the inner product $\langle f, g \rangle = \int_0^4 f(t)g(t)\, dt$. (You may assume this is an inner product.)

a. (10 pts) Find a basis for the vector space P_1 which is orthonormal with respect to that inner product.
b. (10 pts) Find constants a and b which make the quantity $\int_0^2 \left| x^3 - ax - b \right|^2 dx$ as small as possible.

Fall 2017 # 4. Suppose $T : V \to V$ is an invertible linear operator from a vector space V to itself.

a. (10 pts) Show that if $\{v_1, v_2, \ldots v_m\}$ is a linearly independent set in V, then $\{Tv_1, Tv_2, \ldots Tv_m\}$ is also linearly independent.
b. (10 pts) Show by example that at least for some non-zero, non-invertible linear operators, the conclusion of part **a** may fail.

Fall 2017 # 5. Let (x, y) and (a, b) represent vectors in $\mathbb{R}^2$.

a. (10 pts) For each of the following decide whether the formula given defines a norm on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.
(i) $\| (x, y) \|_{ai} = 2 \, | y |$
(ii) $\| (x, y) \|_{aii} = \sqrt{x^4 + y^4}$

b. (10 pts) For each of the following decide whether the formula given defines an inner product on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.
(i) $\langle (a, b), (x, y) \rangle_{bi} = 2ax + 3by$
(ii) $\langle (a, b), (x, y) \rangle_{bii} = 2ay + 3bx$

Fall 2017 # 6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = |\, x \,|$ for $-\pi \leq x \leq \pi$, and extending so that f is 2π-periodic.

a. (7 pts) Compute the Fourier series for f (trigonometric or exponential, your choice).

b. (6 pts) Give a statement of any form of the Parseval identity.

c. (7 pts) Show that $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$.

Here is a small integral table you may use:

$$\int t \sin(at)\, dt = \frac{1}{a^2} \sin(at) - \frac{t}{a} \cos(at)$$

$$\int t \cos(at)\, dt = \frac{1}{a^2} \cos(at) + \frac{t}{a} \sin(at)$$

$$\int t e^{at}\, dt = (at - 1) \frac{e^{at}}{a^2}$$

Fall 2017 # 7. Suppose $\mathcal{H}$ is an inner product space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Suppose $\mathcal{E} = \{e_1, e_2, e_3, \dots\}$ is an orthonormal family in $\mathcal{H}$, and define a function $T : \mathcal{H} \to \mathcal{H}$ by

$$Tf = 3 \langle f, e_1 \rangle e_2 + 5 \langle f, e_2 \rangle e_3.$$

for each f in $\mathcal{H}$.

a. (4 pts) Give a statement of the Cauchy Schwarz inequality.

b. (6 pts) Show that T is a linear operator from $\mathcal{H}$ into $\mathcal{H}$.

c. (10 pts) Show that T is bounded as a linear operator from $\mathcal{H}$ into $\mathcal{H}$. (with the associated norm used on $\mathcal{H}$)

End of Exam