California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Linear Analysis   Fall 2019
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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.
Please
   (1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
   (2) Write on one side of the paper only
   (3) Begin each problem on a new page
   (4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\qquad$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$ $\qquad$ $2 \cos a \cos b = \cos(a-b) + \cos(a+b)$

$2 \sin a \cos b = \sin(a+b) + \sin(a-b)$ $\qquad$ $2 \cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

SOLUTIONS Hoffman

Fall 2019 # 1. Let $\mathcal{V}$ be the space of all continuous real valued 2π-periodic functions on $\mathbb{R}$ for which f' exists and is continuous. For f in $\mathcal{V}$, put

$$\| f \|_a = \int_{-\pi}^{\pi} | f'(t) | \, dt \quad \text{and} \quad \| f \|_b = | f(0) | + \int_{-\pi}^{\pi} | f'(t) | \, dt$$

a. Show that $\| \cdot \|_a$ is not a norm on $\mathcal{V}$.

b. Show that $\| \cdot \|_b$ is a norm on $\mathcal{V}$

You may assume that $\| f \|_1 = \int_{-\pi}^{\pi} | f(t) | \, dt$ does give norm on $\mathcal{V}$.

Solution: **a.**   If f is a constant function on $\mathbb{R}$, then it is in $\mathcal{V}$, and $f'(t) = 0$ for all t. For such a function we have $\| f \|_a = \int_{-\pi}^{\pi} | f'(t) | \, dt = 0$. There are many non-zero constants, so $\| \cdot \|_a$ is not an inner product on $\mathcal{V}$.

    b.   Notice that $\| f \|_b = | f(0) | + \| f' \|_1$. We can use the properties of the known norm, $\| \cdot \|_1$ to help obtain the required properties of the proposed norm $\| \cdot \|_b$ To do this, let f and g be in $\mathcal{V}$, and λ be in $\mathbb{R}$.

    i. $\| f \|_b \geq 0$: $\| f \|_b = | f(0) | + \| f' \|_1 \geq 0$ since each term is nonnegative.

    ii. $\underline{\| f \|_b = 0} \implies f = 0$:   If $\| f \|_b = | f(0) | + \| f' \|_1 = 0$, then $| f(0) | = 0$ and $\| f' \|_1 = 0$ since each term is nonnegative.

      Since $\| f \cdot \|_1$ is a norm,

$$\| f' \|_1 = 0 \implies f' = 0 \implies f \text{ is constant.}$$

      But $| f(0) | = 0 \implies f(0) = 0$. Since f is constant, it must be the zero function as required.

    iii. $\| \lambda f \|_b = | \lambda | \| f \|_b$:   Compute using properties of $\| \cdot \|_1$

$$\begin{aligned} \| \lambda f \|_b &= | (\lambda f)(0) | + \| (\lambda f)' \|_1 = | \lambda f(0) | + \| \lambda f' \|_1 \\ &= | \lambda | | f(0) | + | \lambda | \| f' \|_1 = | \lambda | (| f(0) | + \| f' \|_1) = | \lambda | \| f \|_b \end{aligned}$$

      as required.

    iv. $\| f + g \|_b \leq \| f \|_b + \| g \|_b$:   Compute using properties of $\| \cdot \|_1$

$$\begin{aligned} \| f + g \|_b &= | (f + g)(0) | + \| (f + g)' \|_1 = | f(0) + g(0) | + \| f' + g' \|_1 \\ &\leq (| f(0) | + | g(0) |) + \| f' + g' \|_1 \\ &\leq (| f(0) | + | g(0) |) + (\| f' \|_1 + \| g' \|_1) \\ &= (| f(0) | + \| f \|_1) + (| g(0) | + \| g \|_1) = \| f \|_b + \| g \|_b \end{aligned}$$

The required properties of a norm hold, so this is a norm on $\mathcal{V}$ as claimed.

Fall 2019 # 2.   For $-\pi \le t \le \pi$, define $f(t)$ by $f(t) = \begin{cases} 0, & \text{for } -\pi \le t \le 0 \\ 1, & \text{for } 0 < t \le \pi \end{cases}$.

a.   Compute either the trigonometric or the exponential Fourier series for f. (Your choice which)

b.   Use the result of part **(a)** to show that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

Solution: **a.**   The exponential Fourier series is the L^2-expansion of f with respect to the orthonormal basis given by $e_n(t) = e^{int}$ using the inner product $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$.

$$f(t) \sim \sum_{-\infty}^{\infty} \langle f, e_n \rangle e_n(t) = \sum_{-\infty}^{\infty} \hat{f}(n) e^{int}.$$

We compute

$$\langle f, e_n \rangle = \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt = \frac{1}{2\pi} \int_{0}^{\pi} e^{-int}\, dt$$

$$= \begin{cases} \frac{1}{2}, & \text{if } n = 0 \\ -\frac{1}{2\pi n i} e^{-int} \big|_{t=0}^{\pi}, & \text{if } n \ne 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & \text{if } n = 0 \\ -\frac{1}{2\pi n i} e^{-int}((-1)^n - 1), & \text{if } n \ne 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & \text{if } n = 0 \\ 0, & \text{if } n \text{ is even and } n \ne 0 \\ \frac{1}{\pi n i}, & \text{if } n \text{ is odd} \end{cases}$$

.

The exponential Fourier series is thus

$$\boxed{f(t) \sim \sum_{-\infty}^{\infty} \hat{f}(n) e^{int} = \sum_{k=0}^{\infty} \frac{-1}{(2k+1)\pi i} e^{-(2k+1)it} + \frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)\pi i} e^{(2k+1)it}.}$$

The trigonometric series may be found by combining the terms of the exponential series.

$$f(t) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)\pi i} \left(e^{(2k+1)it} - e^{-(2k+1)it} \right)$$

$$= \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin(2k+1)t$$

This may also be found by writing

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\pi} \int_{0}^{\pi} dt = 1$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{0}^{\pi} \cos nt \, dt = \frac{1}{n\pi} \sin nt \bigg|_{t=0}^{\pi} = 0, \quad n > 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{0}^{\pi} \sin nt \, dt = \frac{-1}{n\pi} \cos nt \bigg|_{t=0}^{\pi}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

The trigonometric Fourier series is

$$\boxed{f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin(2k+1)x}$$

b.   Using the inner product $\langle f, g \rangle = (1/(2\pi)) \int_{-\pi}^{\pi} f(t)\overline{g(t)} \, dt$ and the orthonormal basis given by $e_n(t) = e^{int}$ for $n \in \mathbb{Z}$, we have

$$\| f \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(t) |^2 \, dt = \frac{1}{2\pi} \int_{0}^{\pi} dt = \frac{1}{2}.$$

The Parseval identity becomes

$$\frac{1}{2} = \| f \|^2 = \sum_{-\infty}^{\infty} |\langle f, e_n \rangle|^2 = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pi^2} + \frac{1}{4} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pi^2}.$$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi^2}{8}$$

as claimed. ∎

Fall 2019 # 3.   Let the linear operator K be defined for f in the space $C([0,1])$ of continuous real valued functions on [01] by $(Kf)(x) = \lambda \int_0^x e^{-t} f(t)\,dt$. (Assume that this does give a linear operator from $C([0,1])$ into $C([0,1])$.)

a.   With the norm on $\mathcal{C}([0,1])$ defined by $\| f \|_\infty = \sup\{|f(t)| \mid t \in [0,1]\}$, find a range of values for the parameter λ for which the operator norm of K is smaller than 1.

b.   Describe the iterative process for solving the integral equation

$$f(x) = 1 + \lambda \int_0^x e^{-t} f(t)\,dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first two iterates, $f_1(x)$ and $f_2(x)$.

Solution: **a.**   If $f \in C([0,1])$ and $x \in [0,1]$, then

$$|(Kf)(x)| = \left| \lambda \int_0^x e^{-t} f(t)\,dt \right| = |\lambda| \left| \int_0^x e^{-t} f(t)\,dt \right|$$

$$\leq |\lambda| \int_0^x e^{-t} |f(t)|\,dt \leq |\lambda| \int_0^x e^{-t} \| f \|_\infty \,dt$$

$$\leq |\lambda| \| f \|_\infty \, (-e^{-t})\big|_{t=0}^{x} = |\lambda| \| f \|_\infty \, (-e^{-x} + 1)$$

$$\leq |\lambda| \| f \|_\infty \left(1 - \frac{1}{e} \right) = |\lambda| \| f \|_\infty \left(\frac{e-1}{e} \right).$$

This is true for all x in $[0,1]$, so

$$\| Kf \|_\infty \leq |\lambda| \| f \|_\infty \left(\frac{e-1}{e} \right)$$

For the operator norm, this says that

$$\| K \| \leq |\lambda| \left(\frac{e-1}{e} \right)$$

We have $\| K \| < 1$ provided $|\lambda| < e/(e-1)$.

b.   A solution to the equation $f(x) = 1 + \lambda \int_0^x e^{-t} f(t)\,dt = 1 + \lambda(Kf)(x)$ is a fixed point for the nonlinear transformation $Tf = 1 + Kf$. Since

$$\| Tf - Tg \|_\infty = \| (1 + Kf) - (1 + Kg) \|_\infty = \| Kf - Kg \|_\infty = \| K(f-g) \|_\infty$$
$$\leq \| K \| \| f - g \|_\infty,$$

the transformation T will be a (proper) contraction provided $\| K \| < 1$. We know from part **(a)** that this happen if $|\lambda| < e/(e-1)$. In this case, T is a proper contraction with respect to the metric comoing from the norm supremum norm on the space $C([0,1])$. That space is complete with respect to that metric. (A uniform limit of continuous functions is continuous.)(More precisely, a uniformly Cauchy sequence of continuous functions into a complete space such as $\mathbb{R}$ or $\mathbb{C}$ must converge uniformly to a limit which must be continuous.) The Banach fixed point theporem (contraction mapping principle) says that if f_0 is any function in $\mathcal{C}([0,1])$, then the sequence of iterates f_0, $f_1 = Tf_o$, $f_2 = Tf_1$, ... converges with respect to the norm $\| \cdot \|_\infty$ to a "fixed point" f. Since convergence with respect

to that norm is the same as uniform convergence on the interval $[0, 1]$, the iterates converge uniformly to a solution.

With $f_0(x) = 0$ for all x, we compute

$$f_1(x) = (Tf_0)(x) = 1 + \lambda \int_0^x e^{-t} 0 \, dt = 1$$

$$f_2(x) = (Tf_1)(x) = 1 + \lambda \int_0^x e^{-t} 1 \, dt = 1 + \lambda \left. \left(-e^{-t}\right)\right|_{t=0}^{x} = 1 + \lambda(1 - e^{-x})$$

Fall 2019 # 4. Let u and v be vectors in a Hilbert space $\mathcal{H}$ with $\|u\| = \|v\| = 1$. Show that $\|u + v\| = 2$ if and only if $u = v$.

(The norm is that associated to the inner product on $\mathcal{H}$.)

(Suggestion: Consider also $\|u - v\|^2$ and its relationship with $\|u + v\|^2$.)

Solution: METHOD ONE: We know that if u and v are any two vectors in an inner product space, then they must satisfy the *parallelogram law*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

For our problem we know that $\|u\| = \|v\| = 1$, so

(*) $$\|u + v\|^2 + \|u - v\|^2 = 4.$$

If $\|u + v\| = 2$, then from $(*)$ we can compute

$$4 + \|u - v\|^2 = 4$$
$$\|u - v\|^2 = 0$$
$$\|u - v\| = 0$$
$$u - v = 0$$
$$u = v.$$

Conversely, if $u = v$, then

$$\|u + v\| = \|2u\| = 2\|u\| = 2.$$

METHOD TWO: A direct argument winds up reflecting much of the proof of the parallelogram law. Again, we start out knowing that $\|u\| = \|v\| = 1$ and observe that

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle$$
$$= \|u\|^2 - (\langle u, v \rangle + \langle v, u \rangle) + \|v\|^2$$
$$= 2 - (\langle u, v \rangle + \langle v, u \rangle)$$

Now, if $\|u + v\| = 2$, we have

$$4 = \|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$$
$$= \|u\|^2 + (\langle u, v \rangle + \langle v, u \rangle) + \|v\|^2$$
$$= 2 + (\langle u, v \rangle + \langle v, u \rangle)$$

Thus $\langle u, v \rangle + \langle v, u \rangle = 4 - 2 = 2$ and our last observation becomes

$$\|u - v\|^2 = 2 - 2 = 0.$$

From this follows $u = v$ as in Method One.

The converse implication is as in Method One.

Fall 2019 # 5.   Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = (1, 0, 2, 2) \quad \text{and} \quad v_2 = (1, 0, 0, 1).$$

a.   Show that v_1 and v_2 are linearly independent.

b.   Use the Gram-Schmidt process to find an orthonormal basis for $\mathcal{W}$. (Orthogonal with respect to the usual dot product)

c.   Find the vector on $\mathcal{W}$ closest to the vector $v = (3, 1, 2, 1)$.

Solution: **a.**   To show that v_1 and v_2 are linearly independent, supose $av_1 + bv_2 = 0$, then

$$(0, 0, 0, 0) = a(1, 0, 2, 2) + b(1, 0, 0, 1) = (a + b, 0, 2a, 2a + b)$$

This gives four simultaneous equations for the coefficients a and b.

$$\begin{cases} 0 & = a + b \\ 0 & = 0 \\ 0 & = 2a \\ 0 & = 2a + b \end{cases}$$

The third equation gives $a = 0$. Inserting this into either the first or the fourth gives $b = 0$. This shows that v_1 and v_2 are linearly dependent.

b.   Apply the Gram-Schmidt process to the vectors $v_1 = (1, 0, 2, 2)$ and $v_2 = (1, 0, 0, 1)$. First normalize v_1.

$$\| v_1 \| = \sqrt{1 + 0 + 4 + 4} = \sqrt{9} = 3,$$

so we set $e_1 = (1/3)v_1 = (1/3, 0, 2/3, 2/3)$.

Now project v_2 onto the space spanned by e_1 and subtract.

$$\begin{aligned} u_2 &= v_2 - \langle v_2, e_1 \rangle e_1 \\ &= (1, 0, 0, 1) - \langle (1, 0, 0, 1), (1/3, 0, 2/3, 2/3) \rangle (1/3, 0, 2/3, 2/3) \\ &= (1, 0, 0, 1) - \left(\frac{1}{3} + 0 + 0 + \frac{2}{3} \right) (1/3, 0, 2/3, 2/3) \\ &= (1, 0, 0, 1) - (1/3, 0, 2/3, 2/3) \\ &= (2/3, 0, -2/3, 1/3). \end{aligned}$$

We have

$$\| u_2 \| = \sqrt{\frac{4}{9} + 0 + \frac{4}{9} + \frac{1}{9}} = 1.$$

Thus u_2 is already normalized, and we can put $e_2 = u_2$. Our orthonormal basis is

$$\boxed{e_1 = \left(\frac{1}{3}, 0, \frac{2}{3}, \frac{2}{3} \right) \quad ; \quad e_2 = \left(\frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{3} \right).}$$

c.   The desired closest vector is the projection

$$\begin{aligned} w = Pv &= \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 \\ &= \left\langle (3, 1, 2, 1), \left(\frac{1}{3}, 0, \frac{2}{3}, \frac{2}{3} \right) \right\rangle \left(\frac{1}{3}, 0, \frac{2}{3}, \frac{2}{3} \right) \\ &\quad + \left\langle (3, 1, 2, 1), \left(\frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{3} \right) \right\rangle \left(\frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{3} \right) \end{aligned}$$

$$= \left(1 + 0 + \frac{4}{3} + \frac{2}{3}\right)\left(\frac{1}{3}, 0, \frac{2}{3}, \frac{2}{3}\right) + \left(2 + 0 - \frac{4}{3} + \frac{1}{3}\right)\left(\frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{3}\right)$$

$$= 3\left(\frac{1}{3}, 0, \frac{2}{3}, \frac{2}{3}\right) + 1\left(\frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{3}\right)$$

$$= (1, 0, 2, 2) + \left(\frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{3}\right) = \boxed{\left(\frac{5}{3}, 0, \frac{4}{3}, \frac{7}{3}\right)}$$

Fall 2019 # 6.   For each of the following decide if it is a vector subspace0f the space of all real valued functions on the interval $[0, 1]$. Give reasons for your answers.

    a. $A =$ The set of all polynomials in one variable with real coefficients having degree no more than 2 and the coefficient of x equal to 3.

    b. $B =$ The set of continuous real valued functions on the interval $[0, 1]$ such that $\int_0^1 f(t)\, dt = 0$.

    c. $C =$ The set of all real valued functions on the interval $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$

Solution: **a.** $\boxed{\text{NO}}$ The set A is not a vector space. Any one of at least three reasons is sufficient.

    i. The zero polynomial is not in the set A.

    ii. The set A is not closed under polynomial addition. The polynomials given by $p(x) = x^2 + 3x$ and $q(x) = 3x + 1$ are in A, but $(p + q)(x) = x^2 + 6x + 1$. Since $6 \neq 1$, $p + q \notin A$.

    iii. The set A is not closed under scalar multiplication. $p(x) = x^2 + 3x$ is in A, but $(2p)(x) = 2x^2 + 6x$. Since $6 \neq 1$, $2p \notin A$

   b. $\boxed{\text{YES}}$ The set B is a subset of the vector space of all real valued functions on $[0, 1]$. Furthermore,

    i. The zero function is continuous and $\int_0^1 0\, dt = 0$, so it is in B.

    ii. If f and g are in B, they are continuous, so $f + g$ is also continuous.

$$\int_0^1 (f + g)(t)\, dt = \int_0^1 (f(t) + g(t))\, dt = \int_0^1 f(t)\, dt + \int_0^1 g(t)\, dt = 0 + 0 = 0$$

    Thus $f + g \in B$.

    iii. If $f \in B$ and $\lambda \in \mathbb{R}$, then λf is continuous. Also,

$$\int_0^1 (\lambda f)(t)\, dt = \int_0^1 \lambda f(t)\, dt = \lambda \int_0^1 f(t)\, dt = \lambda \cdot 0 = 0$$

    Thus $\lambda f \in B$.

These show that B is a vector subspace of the vector space of all real valued functions on $[0, 1]$, and so is itself a vector space.

   c. $\boxed{\text{NO}}$ The set C is not a vector space. Any one of at least three reasons is sufficient.

    i. The zero function is not in the set C since its value at 1 is not 1.

    ii. The set C is not closed under polynomial addition. If f and g are in C, then $(f + g)(1) = f(1) + g(1) = 1 + 1 = 2 \neq 1$, so $f + g \notin C$.

    iii. The set C is not closed under scalar multiplication. If f is in C, then $(2f)(1) = 2f(1) = 2 \cdot 1 = 2 \neq 1$, so $2f \notin C$.

Fall 2019 # 7. Each of the following is a function from the vector space $C([0,1], \mathbb{R})$ of all continuous real valued functions on $[0,1]$ to $\mathbb{R}$. For each, decide whether it is linear. If it is, prove it. If it is not, show by an example or an explanation that it is not.

    a. $A : C([0,1], \mathbb{R}) \to \mathbb{R}$ given by $A(f) = f(1) + \int_0^1 f(t)e^t \, dt$.

    b. $B : C([0,1], \mathbb{R}) \to \mathbb{R}$ given by $B(f) = \int_0^1 |f(t)| \, dt$.

Solution: **a.** $\boxed{\text{YES}}$ Suppose f and g are in $C([0,1], \mathbb{R})$ and λ and μ in $\mathbb{R}$. Then

$$A(\lambda f + \mu g) = (\lambda f + \mu g)(1) + \int_0^1 (\lambda f + \mu g)(t)e^t \, dt$$

$$= (\lambda f(1) + \mu g(1)) + \int_0^1 (\lambda f(t) + \mu g(t))e^t \, dt$$

$$= (\lambda f(1) + \mu g(1)) + \int_0^1 \lambda f(t)e^t \, dt + \int_0^1 \mu g(t)e^t \, dt$$

$$= \left(\lambda f(1) + \int_0^1 \lambda f(t)e^t \, dt \right) + \left(\mu g(1) + \int_0^1 \mu g(t)e^t \, dt \right)$$

$$= \left(\lambda f(1) + \lambda \int_0^1 f(t)e^t \, dt \right) + \left(\mu g(1) + \mu \int_0^1 g(t)e^t \, dt \right)$$

$$= \lambda \left(f(1) + \int_0^1 f(t)e^t \, dt \right) + \mu \left(g(1) + \int_0^1 g(t)e^t \, dt \right)$$

$$= \lambda A(f) + \mu A(g) \qquad \text{as required for linearity.}$$

    b. $\boxed{\text{NO}}$ This transformation is neither additive nor homogeneous and is not linear. In particular, for linearity we should have $B(\lambda f) = \lambda B(f)$. But

$$B(\lambda f) = \int_0^1 |(\lambda f)(t)| \, dt = \int_0^1 |\lambda| \cdot |f(t)| \, dt = |\lambda| \int_0^1 |f(t)| \, dt = |\lambda| B(f).$$

If $B(f) \neq 0$ and λ is negative, then this is not equal to $\lambda B(f)$. thus the function B is not linear.