California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination

Linear Analysis Spring 2001

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Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a, b])$ denotes the space of all continuous functions on the interval $[a, b]$. If there is need to specify the possible values, $\mathcal{C}([a, b], \mathbb{R})$ will denote the space of all continuous real valued functions on $[a, b]$ and $\mathcal{C}([a, b], \mathbb{C})$ the space of all continuous complex valued functions.
$L^2([a, b])$ denotes the space of all functions on the inteval $[a, b]$ such that $\int_a^b |f(x)|^2 \, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\, dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Spring 2001 # 1. **a.** Show that the family

$$\mathcal{T} = \{1/\sqrt{2\pi}, (1/\sqrt{\pi})\cos nx, (1/\sqrt{\pi})\sin nx : n = 1, 2, 3, \dots\}$$

of functions on the interval $[-\pi, \pi]$ is orthonormal with respect to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

b. Find the Fourier series for the function $f(x) = x$ on $[-\pi, \pi]$ using the family $\mathcal{T}$.

c. Discuss briefly what it means for an orthonormal family to be a complete orthonormal family or an orthonormal basis and how you know that $\mathcal{T}$ is one.

d. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Spring 2001 # 2. For each continuous function f on the interval $[0, 2]$ define a function Tf by

$$(Tf)(x) = x + \lambda \int_0^x xtf(t)\, dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0, 2])$ with respect to the supremum norm. Justify your answer.

b. Describe the iterative process for solving the integral equation

$$f(x) = x + \lambda \int_0^x xtf(t)\, dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

c. Show that if f is a solution to the integral equation of part (c), then it is also a solution to the differential equation

$$f''(x) - \lambda x^2 f'(x) - 3\lambda x f(x) = 0 \qquad \text{with } f(0) = 0 \text{ and } f'(0) = 1.$$

Spring 2001 # 3. Let f and g be vectors in a Hilbert space $\mathcal{H}$.

a. Show that $\| f + g \|^2 + \| f - g \|^2 = 2\| f \|^2 + 2\| g \|^2$.

b. Show that $\| f + g \| \cdot \| f - g \| \leq \| f \|^2 + \| g \|^2$.

Spring 2001 # 4. **a.** Define what it means for a set $\mathcal{L}$ of vectors in a vector space $\mathcal{V}$ to be linearly independent. (Be careful. Your definition should work for a set $\mathcal{L}$ which might be infinite.)

b. Show that the vectors

$$\vec{u} = (1, 2, 3, 2) \quad , \quad \vec{v} = (4, 3, 2, 1) \quad , \quad \vec{w} = (-7, 1, 9, 7)$$

in $\mathbb{R}^4$ are linearly dependent (not independent).

b. Show that the set of functions $\mathcal{E} = \{e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int} : n = 0, \pm 1, \pm 2, \pm 3 \ldots\}$ is linearly independent as a set of functions on $[-\pi, \pi]$ (vectors in an approipriate function space.)

Spring 2001 # 5. **a.** Let φ be a continuous function on the interval $[a, b]$. Define $M_\varphi : L^2([a, b]) \to L^2([a, b])$ by $(M_\varphi f)(x) = \varphi(x) f(x)$.

Show that M_φ is a bounded linear operator on $L^2([a, b])$.

b. Let $k(x, t)$ be a continuous function on the square $[a, b] \times [a, b]$. Define $K : L^2([a, b]) \to L^2([a, b])$ by $(Kf)(x) = \int_a^b k(x, t) f(t)\, dt$.

Show that K is a bounded linear operator on $L^2([a, b])$.

Spring 2001 # 6. Suppose $\mathcal{E} = \{e_1, e_2, e_3, \ldots\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$. Let $\mathcal{M}_n$ be the vector subspace span$(\{e_1, e_2, \ldots, e_n\})$ spanned by the first n basis vectors. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be numbers. For v in $\mathcal{H}$, put $Tv = \sum_{k=1}^n \lambda_k \langle v, e_k \rangle e_k$

a. Show that T is a linear operator from $\mathcal{H}$ into $\mathcal{H}$.

b Show that T is a bounded linear operator.

c. Show that the operator norm of T is $\| T \| = \max(\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\})$.

Spring 2001 # 7. For each continuous function f on the interval $[0, 1]$, define a function Kf on $[0, 1]$ by

$$(Kf)(x) = \int_0^1 x^2 t^2 f(t)\, dt = x^2 \int_0^1 t^2 f(t)\, dt.$$

a. Find any nonzero eigenvalues for the operator K and the associated eigenvectors.

b. For given function g, consider the integral equation

$$f(x) = g(x) + \lambda \int_0^1 x^2 t^2 f(t)\, dt.$$

Use a Neumann series to solve this equation when $g(x) = x$.

c. Find a function $R(x, t, \lambda)$ such that solutions to the integral equation of part (b) are given by

$$f(x) = g(x) + \lambda \int_0^1 R(x, t, \lambda) g(t)\, dt.$$

Spring 2001 # 8. Consider the boundary value problem

$$-\frac{d}{dx}\left(\frac{df}{dx}\right) = \phi(x) \qquad \text{for } 0 \le x \le 1 \qquad \text{with } f(0) = f'(0) \text{ and } f(1) = f'(1).$$

a. Find a function $G(x,t)$ such that solutions to the problem are given by

$$f(x) = \int_0^1 G(x,t)\phi(t)\,dt.$$

b. Use the result of part **(a)** to solve the problem finding $f(x)$ when $\phi(x) = x$ for all x.

End of Exam