California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Spring 2006

Cooper, Gutarts, Hoffman*

Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.
$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\,dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Spring 2006 # 1. **a.** Give a statement of any form of the Parseval identity. (Make sure you explain the things that appear in your statement.)

b. Let $f(x) = x$ for $-\pi < x \le \pi$ and consider f to be extended as a 2π-periodic function on $\mathbb{R}$. Compute the Fourier series for f. (Either exponential or trigonometric. Your choice.)

c. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Spring 2006 # 2. Which of the following are vector subspaces of the vector space $\mathcal{F}([0,1]), \mathbb{R})$ of all real valued functions on the interval $[0,1]$? For any which are, prove it. For any which are not, explain how you know they are not.

a. All polynomials with real coefficients of degree exactly 3.
b. All bounded integrable functions on $[0,1]$ with $f(0) = 0$
c. All continuous real valued functions on $[0,1]$ which are differentiable for $0 < x < 1$ with $f'(x) + f(x) = 1$.
d. All polynomials with real coefficients of degree no more than 3, including the zero polynomial.

Spring 2006 # 3. **a.** Suppose $\langle \cdot, \cdot \rangle$ is an inner product on a vector space $\mathcal{V}$, and $\| \cdot \|$ is the associated norm. Show that $\| v + w \|^2 + \| v - w \|^2 = 2 \| v \|^2 + 2 \| w \|^2$ for all v and w in $\mathcal{V}$.

(Simply saying that you know it is true will not suffice. You must give a proof.)

b. Show that the formula $\| (x, y) \| = |x| + |y|$ gives a norm on $\mathbb{R}^2$.

c. Show that there is no possible inner product on $\mathbb{R}^2$ for which the norm of part **b** is the associated norm.

Spring 2006 # 4. Let $\mathcal{P}_2$ be the space of all polynomials of degree no more than 2 (including the zero polynomial) considered as functions on the interval $[0,2]$. For p and q in $\mathcal{P}_2$, let $\langle p, q \rangle = \int_0^2 p(x)\overline{q(x)}\, dx$, and let $\| \cdot \|_2$ be the norm associated with this inner product. Let $\| p \|_\infty = \sup\{|p(x)| \mid x \in [0,2]\}$.

For p in $\mathcal{P}_2$, let $\varphi(p) = p(1)$.

a) Show φ is a linear functional on $\mathcal{P}_2$.
b) Show φ is continuous with respect to the norm $\| \cdot \|_\infty$ on $\mathcal{P}_2$.

(You may do parts (c) and (d) in either order.) (Depending on what you knnow, it might be easier to do d. first)

c) Show φ is continuous with respect to the norm $\| \cdot \|_2$ on $\mathcal{P}_2$.
d) Find a polynomial $q(x)$ in $\mathcal{P}_2$ such that $\varphi(p) = \langle p, q \rangle$ for every p in $\mathcal{P}_2$.

Spring 2006 # 5. **a)** Find a basis for the space $\mathcal{P}_1$ of all polynomials of degree no more than 1 (including the zero polynomial) which is orthonormal with respect to the inner product $\langle p, q \rangle = \int_0^2 p(t)\overline{q(t)}\, dt$.

b) Use the result of part **(a)** to find constants a and b which minimize the quantity $J(a,b) = \int_0^2 \left| t^2 - a - bt \right|^2 \, dt$

Spring 2006 # 6. For each continuous function f on the interval $[0, 1]$, define a function Tf by

$$(Tf)(x) = 1 + \lambda \int_0^x xtf(t)\,dt = 1 + \lambda x \int_0^x tf(t)\,dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm. Justify your answer.

b. Describe the iterative process for solving the integral equation

$$f(x) = 1 + \lambda \int_0^x xtf(t)\,dt \tag{*}$$

specifying the transformation to be iterated and explaing how this leads to a solution. With $f_0(x) = 1$ for all x as the starting function, compute the first two iterates, $f_1(x)$ and $f_2(x)$.

c. Show that a solution to the equation $(*)$ is also a solution to the initial value problem

$$f''(x) = 3\lambda x f(x) + \lambda x^2 f'(x) \quad ; f(0) = 1, f'(0) = 0 \tag{**}$$

Spring 2006 # 7. Find a continuous function f on the interval $[0, 1]$ such that

$$f(x) = x^2 + 2 \int_0^1 (x - t)f(t)\,dt$$

End of Exam