California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Spring 2009

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(Corrected Oct. 2009 MJH)

Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

1. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
2. Write on one side of the paper only
3. Begin each problem on a new page
4. Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|\, z\,|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |\, f(x)\,|^2 \, dx < \infty$

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\qquad$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$ $\qquad$ $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$ $\qquad$ $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Spring 2009 # 1. (Corrected Oct. 2009 MJH) Let $\mathcal{M}$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = (1,0,0,0) \qquad v_2 = (1,0,1,0) \quad \text{and } v_3 = (0,1,0,1).$$

a. Find a basis for $\mathcal{M}$ which is orthonormal with respect to the usual inner product (dot product) on $\mathbb{R}^4$.

b. Find the vector w in $\mathcal{M}$ at minimum distance from $w_o = (1,1,0,0)$.

Spring 2009 # 2. Consider the following formulas for $v = (x, y)$ and $w = (a, b)$ in $\mathbb{R}^2$.

a. Decide for each of the following whether it defines norm on $\mathbb{R}^2$. If "yes", prove it. If "no", show that it is not.

(i) $\| v \|_i = x^2 + y^2$ **(ii)** $\| v \|_{ii} = |x| + 2|y|$

b. Decide for each of the following whether it defines an inner product on $\mathbb{R}^2$. If "yes", prove it. If "no", show that it is not.

(i) $\langle v, w \rangle_i = 3xa - 5yb$ **(ii)** $\langle v, w \rangle_{ii} = xb + ya$

Spring 2009 # 3. Let $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < \pi \\ -1, & \text{for } -\pi < x < 0 \\ 0 & \text{for } x = -\pi, 0, \pi \end{cases}$$

and extending 2π-periodically.

a. Find the Fourier series for $f(x)$. (Exponential form or trigonometric form, your choice)

b. Use the result of part **a** to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

Spring 2009 # 4. **a.** Suppose $\mathcal{L}$ is an orthonormal family of vectors in a Hilbert space $\mathcal{H}$. Show that $\mathcal{L}$ is a linearly independent subset of $\mathcal{H}$.

b. For each positive integer k, let $e_k(t) = e^{ikt}$. Show that the set $\mathcal{E} = \{e_1, e_2, e_3, \ldots\}$ is linearly independent as a set of functions in the space $C([-\pi, \pi])$ of continuous functions functions on the interval $[a, b]$

Spring 2009 # 5. **a.** Describe the Neumann series method for inversion of a bounded linear operator of the form $I - S$ on a Banach space $\mathcal{X}$. Be sure to include conditions sufficient to ensure convergence of the method.

b. Let T be the linear operator on $\mathbb{R}^3$ represented with respect to the standard orthonormal basis by the matrix $\begin{pmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$. Use the method of part **a** to find T^{-1}.

Spring 2009 # 6. For each continuous function f on the interval $[0,2]$ define a function Tf by

$$(Tf)(x) = 1 + \lambda \int_0^x (x^2 - t^2) f(t)\,dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0,2])$ with respect to the supremum norm. Justify your answer.

b. Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0,2])$ with respect to the L^2 norm on $C([0,2])$. Justify your answer.

c. Describe the iterative process for solving the integral equation

$$f(x) = 1 + \lambda \int_0^x (x^2 - t^2) f(t)\,dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

Spring 2009 # 7. Let $\mathcal{V}$ be the space $L^2[-\pi,\pi])$ with the inner product $\langle f, g \rangle = (1/2\pi) \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$.

For integers k, let $e_k(t) = e^{ikt}$.

Consider the operator K on $\mathcal{V}$ given by

$$(Kf)(x) = \int_{-\pi}^{\pi} \cos(x-t) f(t)\,dt.$$

- **a.** Show that each e_k is an eigenvector for K and find the corresponding eigenvalues. (Suggestion: You might want to write the cosine in terms of exponentials.)
- **b.** Find a function $R(x,t,\lambda)$ such that solutions f to the equation $f = g + \lambda K f$ are given by

$$f(x) = g(x) + \lambda \int_{-\pi}^{\pi} R(x,t,\lambda) g(t)\,dt.$$

For what values of λ (if any) does your method fail?

End of Exam