California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Spring 2015
Gutarts*, Hoffman, Krebs

Do five **of the following seven problems. All problems count equally. If you attempt more than five, the best fivewill be used.**

Please

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\qquad$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$ $\qquad$ $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$ $\qquad$ $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$\displaystyle\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$ $\qquad$ $\displaystyle\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$

$\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$ $\qquad$ $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$\displaystyle\int x\sin bx\,dx = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx$ $\qquad$ $\displaystyle\int x\cos bx\,dx = \frac{1}{b^2}\cos bx + \frac{x}{b}\sin bx$

Spring 2015 # 1. For each of the following decide if it is a vector space over $\mathbb{R}$. Give reasons for your answers. (You may assume that the set of all real valued functions on the interval $[0,1]$ is a vector space with the operations $(f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda f(x)$.)

- **a.** $A =$ The set of all polynomials in one variable with real coefficients having degree no more than 2 and the coefficient of x equal to 1
- **b.** $B =$ The set of continuous real valued functions on the interval $[0,1]$ such that $f(0) = 0$ and $f(1) = 0$
- **c.** $C =$ The set of continuous real valued functions on the interval $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$

Spring 2015 # 2. Suppose $T : V \to V$ is an invertible linear operator from a vector space V to itself.

- **a.** Show that if $\{v_1, v_2, \ldots v_m\}$ is a linearly independent set in V, then $\{Tv_1, Tv_2, \ldots Tv_m\}$ is also linearly independent.
- **b.** Show by example that at least for some non-zero, non-invertible linear operators, the conclusion of part **a** may fail.

Spring 2015 # 3. Suppose $\langle \cdot, \cdot \rangle$ is an inner product on a space $\mathcal{H}$ and $\| \cdot \|$ is the associated norm. Let w be in $\mathcal{H}$ and $\{v_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ such that $\lim\limits_{n\to\infty} \| v_n \| = \| w \|$.

- **a.** Show that if we also have $\lim\limits_{n\to\infty} \langle v_n, w \rangle = \langle v, w \rangle$, then $\lim\limits_{n\to\infty} v_n = w$. (Suggestion: you might look at $\| v_n - w \|^2$.)
- **b.** Is the conclusion of part **a**, $\lim\limits_{n\to\infty} v_n = w$, necessarily true without the hypothesis that $\lim\limits_{n\to\infty} \langle v_n, w \rangle = \langle v, w \rangle$? Prove or give a counterexample.

Spring 2015 # 4. In each of the following, T is a function from the vector space $C([0,1],\mathbb{R})$ of all continuous real valued functions on $[0,1]$ to $\mathbb{R}$. For each, decide whether T is linear. If it is, prove it. If it is not, show by an example or an explanation that it is not.

- **a.** $T : C([0,1],\mathbb{R}) \to \mathbb{R}$ given by $T(f) = f(0) + \int_0^1 f(t)e^t\, dt$.
- **b.** $T : C([0,1],\mathbb{R}) \to \mathbb{R}$ given by $T(f) = f(0) + (f(1))^2$.

Spring 2015 # 5. Let $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < \pi \\ -1, & \text{for } -\pi < x < 0 \\ 0 & \text{for } x = -\pi, 0, \pi \end{cases}$$

and extending 2π-periodically.

- **a.** Find the Fourier series for $f(x)$. (Exponential form or trigonometric form, your choice)
- **b.** Use the result of part **a** to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

Spring 2015 # 6. For each continuous function f on the interval $[0, 1]$ and numerical parameter λ define Tf on $[0, 1]$ by

$$(Tf)(x) = x + \lambda \int_0^x f(t)\cos(\pi t)\,dt$$

and consider the integral equation

$$\text{(IE)} \qquad f(x) = x + \lambda \int_0^x f(t)\cos(\pi t)\,dt$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Justify your answer.

b. Describe the iterative process for solving the integral equation above specifying the transformation to be iterated and explaining how and why this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

c. Show that if $f(x)$ is a continuous solution to the integral equation (IE), then it is also a solution to an initial value problem of the form

$$f'(x) + p(x)y(x) = \phi(x) \qquad ; \qquad y(0) = A.$$

Find $p(x)$, $\phi(x)$, and A.

Spring 2015 # 7. Let $C([-\pi, \pi])$ be the space of all continuous real valued functions on the interval $[-\pi, \pi]$ with the norm $\|f\|_\infty = \sup\{|f(t)| : t \in [-\pi, \pi]\}$. (You may assume this is a norm on $C([-\pi, \pi])$).

Let $\phi : C([-\pi, \pi]) \to \mathbb{R}$ be defined by $\phi(f) = \int_{-\pi}^{\pi} f(t)\,dt$

a. Show ϕ is linear on $C([-\pi, \pi])$.

b. Show ϕ is continuous on $C([-\pi, \pi])$.

c. Find the operator norm of ϕ.

End of Exam