CSULA Mathematics
Masters Degree Comprehensive Examination
Linear Analysis Spring 2019
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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Spring 2019 # 1. Let $\mathcal{P}$ be the space of all polynomials with real coefficients. For each of the transformation defined below, decide whether it is linear. For any which are linear, decide whether they are invertible. Be sure to give reasons for your answers. (There is no norm given here. Continuity of the operator or of the inverse is not part of the questions.)

a. $A : \mathcal{P} \to \mathcal{P}$ given by $(Ap)(t) = p'(t)$.
b. $B : \mathcal{P} \to \mathcal{P}$ given by $(Bp)(t) = (p(t))^2$.
c. $C : \mathcal{P} \to \mathcal{P}$ given by $(Cp)(t) = 3p(t)$.

Spring 2019 # 2. Let $T : \mathcal{V} \to \mathcal{W}$ be an invertible linear transformation from a vector space $\mathcal{V}$ onto a vector space $\mathcal{W}$. Show that if $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set in $\mathcal{V}$, then the set $\{Tv_1, Tv_2, \ldots, Tv_n\}$ is a linearly independent set in $\mathcal{W}$.

Spring 2019 # 3. Let $\mathcal{H}$ be an inner product space. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ and g is a vector in $\mathcal{H}$.

a. Show that if $\| f_n \| \to \| g \|$ and $\langle f_n, g \rangle \to \langle g, g \rangle$ as $n \to \infty$, then $f_n \to g$ as $n \to \infty$.
b. Must the conclusion of part **(a)** be true if the hypothesis that $\langle f_n, g \rangle \to \langle g, g \rangle$ is omitted? Justify your answer.

Spring 2019 # 4. Assume that the formula $\langle f, g \rangle = \int_0^1 f(t)g(t)\, t$ gives an inner product on the space of all continuous real valued functions on the interval $[0, 1]$. Let $\mathcal{P}_1$ be the space consisting of all polynomials with real coefficients and degree no more that 1.

a. Find a basis for $\mathcal{P}_1$ which is orthonormal with respect to the inner product given above.
b. Use the result of part **(a)** to find numbers a and b which minimize he quantity $J(a, b) = \int_0^1 (at + b - t^3)^2\, dt$.

Spring 2019 # 5. Let ℓ^2 be the space of all sequences $\vec{x} = (x_1, x_2, x_3, \dots)$ of real numbers such that $\sum_{k=1}^{\infty} x_k^2 < \infty$ with the usual norm and inner product. Let $(a_1, a_2, a_3, \dots)$ be a bounded sequence of real numbers. For $\vec{x}$ in ℓ^2, let $T(\vec{x})$ be the sequence $(a_1x_1, a_2x_2, a_3x_3, \dots)$.

a. Show that T maps ℓ^2 into ℓ^2 and is linear on ℓ^2.

b. Is T bounded as a linear operator on ℓ^2? If it is, then find the operator norm, $\| T \|$. (The answer would be in terms of something having to do with the sequence $\{a_k\})_{k=1}^{\infty}$.

Spring 2019 # 6. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$. Suppose $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator such that $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all f and g in $\mathcal{H}$.

a. Show that all eigenvalues of T are real.

b. Show that eigenvectors of T corresponding to different eigenvalues are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$.

Spring 2019 # 7. For each continuous function f on the interval $[0, 1]$ let Tf be the function on $[0, 1]$ defined by $(Tf)(x) = e^x + \lambda \int_0^x e^{x-t} f(t)\, dt$.

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm given by $\| f \|_\infty = \sup\{| f(t) | : t \in [0, 1]\}$.

b. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the L^2 norm given by $\| f \|_2 = \left(| f(t) |^2 \right)^{1/2}$.

c. Describe the iterative process for solving he integral equation $f(x) = e^x + \lambda \int_0^x e^{x-t} f(t)\, dt$ specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first three iterates $f_1(x)$, $f_2(x)$, and $f_3(x)$.

End of Exam