List backbone colouring of graphs

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Abstract

Suppose G is a graph and H is a subgraph of G. Let L be a mapping that assigns to each vertex v of G a set $L(v)$ of positive integers. We say (G, H) is backbone L-colourable if there is a proper vertex colouring c of G such that $c(v) \in L(v)$ for all $v \in V$, and $|c(u) - c(v)| \geqslant 2$ for every edge uv in H. We say (G, H) is backbone k-choosable if (G, H) is backbone L-colourable for any list assignment L with $|L(v)| = k$ for all $v \in V(G)$. The backbone choice number of (G, H), denoted by $\mathrm{ch}_{\mathrm{BB}}(G, H)$, is the minimum k such that (G, H) is backbone k-choosable. The concept of backbone choice number is a generalization of both the choice number, and the $L(2, 1)$-choice number. Precisely, if $E(H) = \emptyset$ then $\mathrm{ch}_{\mathrm{BB}}(G, H) = \mathrm{ch}(G)$, where $\mathrm{ch}(G)$ is the choice number of G; if $G = H^2$ then $\mathrm{ch}_{\mathrm{BB}}(G, H)$ is the same as the $L(2, 1)$-choice number of H. In this article, we first show that if $|L(v)| = d_G(v) + 2d_H(v)$ then (G, H) is L-colourable, unless $E(H) = \emptyset$ and each block of G is a complete graph or an odd cycle. This generalizes a result of Erdős, Rubin and Taylor on degree-choosable graphs. Secondly, we prove that $\mathrm{ch}_{\mathrm{BB}}(G, H) \leqslant \max\{\lfloor \mathrm{mad}(G) \rfloor + 1, \lfloor \mathrm{mad}(G) + 2\mathrm{mad}(H) \rfloor\}$, where $\mathrm{mad}(G)$ is the maximum average degree of a graph G. Finally, we establish various upper bounds on $\mathrm{ch}_{\mathrm{BB}}(G, H)$ in terms of $\mathrm{ch}(G)$. In particular, we prove that for a k-choosable graph G, $\mathrm{ch}_{\mathrm{BB}}(G, H) \leqslant 3k$ if every component of H is unicyclic; $\mathrm{ch}_{\mathrm{BB}}(G, H) \leqslant 2k$ if H is a matching; and $\mathrm{ch}_{\mathrm{BB}}(G, H) \leqslant 2k + 1$ if H is a disjoint union of paths with lengths at most two.

1 Introduction

Backbone colouring of graphs is a model for the channel assignment problem. The task in the channel assignment problem is to assign channels to a set of transmitters such that interference is avoided. We divide interferences into two types, strong and

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weak. The channels assigned to two transmitters with strong interference should be far apart, and the channels assigned to two transmitters with weak interference should be distinct. We construct a graph G whose vertices represent transmitters, and two vertices are adjacent if the two corresponding transmitters interfere with each other. We further mark those edges connecting two vertices representing transmitters with strong interference, and denote by H the subgraph of G induced by the marked edges. The subgraph H is called the *backbone* of G; and (G, H) is called a *graph pair*.

For a graph pair (G, H), a *backbone k-colouring* of (G, H) is a mapping $c : V(G) \to \{1, 2, \dots, k\}$ such that c is a proper colouring of G, i.e., adjacent vertices of G receive distinct colours, and moreover, $|c(u) - c(v)| \geqslant 2$ for every edge uv in H. The *backbone chromatic number of* (G, H), denoted by $\chi_{\mathrm{BB}}(G, H)$, is the minimum k for which there is a backbone k-colouring of (G, H).

Backbone colouring was first introduced by Broersma et al. [1] as a generalization of distance-two labeling (also known as $L(2, 1)$-labeling) which, also motivated by the channel assignment problem, has been studied extensively in the literature (cf. [6, 7, 8, 10, 11, 12, 14, 15, 16, 17, 18, 22, 23, 24]). A distance-two labeling of a graph G is a function f that assigns to each vertex v a non-negative integer $f(v)$ so that $|f(u) - f(v)| \geqslant 2$ if $uv \in E(G)$; and $f(u) \neq f(v)$ if $d_G(u, v) = 2$. By letting $G = H^2$ (that is, $V(G) = V(H)$ and $uv \in E(G)$ if $1 \leqslant d_H(u, v) \leqslant 2$), a backbone colouring of (G, H) is equivalent to a distance two labeling of H.

Backbone colouring of graphs has attracted considerable attention lately (cf. [1, 2, 3, 4, 5, 19]). It is easy to see that for any graph pair (G, H), we have $\chi_{\mathrm{BB}}(G, H) \leqslant 2\chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. This upper bound is tight even if H is a spanning tree of G. It was shown in [2] that for any positive integer n, there exists an n-chromatic graph with a spanning tree T such that $\chi_{\mathrm{BB}}(G, T) = 2n - 1$. The result is strengthened in [19], where it was shown that the graph G can be chosen to be triangle-free. In [5], the result was further strengthened: there exist n-chromatic graphs of arbitrary large girth with a spanning tree T such that $\chi_{\mathrm{BB}}(G, T) = 2n - 1$.

A *list assignment* of a graph G is a mapping L which assigns to each vertex v a set $L(v)$ of positive integers. We say G is *L-colourable* if there exists a proper vertex colouring c of G such that $c(v) \in L(v)$ for every $v \in G$. A graph G is *k-choosable* if G is L-colourable for every list assignment L with $|L(v)| = k$ for all $v \in V(G)$. The *choice number* of G, denoted by $\mathrm{ch}(G)$, is the minimum k such that G is k-choosable.

We investigate the list version of backbone colouring of graphs. Let (G, H) be a graph pair. A *backbone L-colouring* of (G, H) is a backbone colouring c of (G, H) such that $c(v) \in L(v)$ for every vertex v. We say (G, H) is *backbone L-colourable* if there exists a backbone L-colouring of (G, H). Given a mapping $\phi : V(G) \to \{1, 2, \ldots\}$, we say (G, H) is *backbone ϕ-choosable* if for any list assignment L with $|L(v)| = \phi(v)$ for every vertex v, (G, H) is backbone L-colourable. We say (G, H) is *backbone k-choosable* if (G, H) is backbone ϕ-choosable for the constant function $\phi \equiv k$. The *backbone choice number* of (G, H), denoted by $\mathrm{ch}_{\mathrm{BB}}(G, H)$, is the minimum k such that (G, H) is backbone k-choosable. Note that if $E(H) = \emptyset$, then $\mathrm{ch}_{\mathrm{BB}}(G, H) = \mathrm{ch}(G)$. If $G = H^2$, then $\mathrm{ch}_{\mathrm{BB}}(G, H)$ is equivalent to the $L(2, 1)$-choice number of H.

The aim of this article is to generalize several known results concerning the choice number of graphs and the $L(2, 1)$-choice number of graphs to the backbone choice number, and to investigate relations between the backbone choice number of (G, H) with other parameters of graphs G and subgraphs H.

Denote by $d_G(v)$ the degree of v in G. A graph G is called *degree-choosable* if G is L-colourable for every list assignment L with $|L(v)| = d_G(v)$. It was proved by Erdős, Rubin and Taylor [9] that every graph G is degree-choosable, unless each block of G is either a complete graph or an odd cycle. We say a graph pair (G, H) is *backbone degree-choosable* if (G, H) is backbone L-colourable for every list assignment L with $|L(v)| = d_G(v) + 2d_H(v)$. In Section 2, we generalize the above result of Erdős, Rubin and Taylor to list backbone colouring of graphs by proving that for any connected graph G and any spanning subgraph H of G, (G, H) is backbone degree choosable, unless $E(H) = \emptyset$ and G is a graph such that each block is an odd cycle or a complete graph.

The *maximum average degree* of a graph G is defined as $\mathrm{mad}(G) = \max \frac{2|E(G')|}{|V(G')|}$, where the maximum is taken over all subgraphs G' of G. In Section 3, we show that for any graph pair (G, H), $\mathrm{ch}_{\mathrm{BB}}(G, H) \leqslant \max\{\lfloor \mathrm{mad}(G) \rfloor + 1, \lfloor \mathrm{mad}(G) + 2\mathrm{mad}(H) \rfloor\}$. In Section 4, we establish several upper bounds for $\mathrm{ch}_{\mathrm{BB}}(G, H)$ in terms of the choice number of G and the structure of H. In particular, assuming G is k-choosable we prove the following results:

$$\operatorname{ch}_{\mathrm{BB}}(G,H) \leqslant \begin{cases} 2k & \text{if } H \text{ is a matching;} \\ 3k & \text{if each component of } H \text{ is unicyclic;} \\ 2k+1 & \text{if } H \text{ is a disjoint union of paths with length one or two.} \end{cases}$$

2 Degree Choosability

Erdős, Rubin, and Taylor [9] proved the following results.

Theorem 1. [9] *Let G be a connected graph. Then G is degree-choosable, unless each block of G is either an odd cycle or a complete graph.*

We shall extend this result to list backbone colouring of graphs. For this purpose, we need a few more definitions and notations. Let G be a graph with vertex set V. For a subset V' of V, we denote $G[V']$ the subgraph of G induced by V'. Let (G,H) be a graph pair. Assume $V' \subset V(G)$ and c is a backbone colouring of $(G[V'], H[V'])$. Let $w \in V(G) \setminus V'$. Assume c' is a backbone colouring of $(G[V' \cup \{w\}], H[V' \cup \{w\}])$ with $c'(v) = c(v)$ for all $v \in V'$ and $c'(w) = i$. Then we say c' *extends* c, and that i *can be used on w to extend c*. In addition, throughout the article, for a graph G with $E' \subset E(G)$ and $V' \subset V(G)$ we denote $G - E'$ the subgraph of G obtained by deleting the edges in E'; and $G - V'$ the subgraph of G induced by the vertices $V(G) \setminus V'$. In case $V' = \{v\}$, we write $G - v$ for $G - V'$.

Lemma 1. *Assume G is a connected graph, H is a subgraph of G, and ϕ is an integer mapping on $V(G)$. If $\phi(v) \geqslant d_G(v) + 2d_H(v)$ for every $v \in V(G)$ and the inequality is strict for some vertex u, then (G,H) is backbone ϕ-choosable.*

Proof. Order the vertices of G as $v_1, v_2, \ldots, v_n$ so that $u = v_n$ and every vertex v_i with $i < n$ has a neighbour v_j with $j > i$. Such an ordering exists as G is connected. Let L be a ϕ-list assignment of G. We colour the vertices greedily in this order. When we colour v_i for some $i < n$, the number of colours forbidden for v_i, due to its coloured neighbours, is at most $a+3b$, where a and b denote the number of coloured neighbours of v_i in $G-E(H)$ and in H, respectively. As v_i has at least one uncoloured neighbour, we have

$$a + 3b < d_{G-E(H)}(v_i) + 3d_H(v_i) = d_G(v_i) + 2d_H(v_i) \leqslant |L(v_i)|.$$

Therefore there is a colour in $L(v_i)$ that can be used on v_i to extend the current partial colouring. At the last step (when we colour v_n), because $\phi(v_n) > d_G(v_n) + 2d_H(v_n)$, the same argument shows that there is a colour in $L(v_n)$ that can be used on v_n. □

Theorem 2. *Let G be a connected graph and H a spanning subgraph of G. Then (G, H) is backbone degree-choosable, unless $E(H) = \emptyset$ and every block of G is either an odd cycle or a complete graph.*

Proof. Let $\phi(v) = d_G(v) + 2d_H(v)$. Assume L assigns to each vertex v a list $L(v)$ of $\phi(v)$ permissible colours. If $E(H) = \emptyset$, the result follows from Theorem 1. Suppose $xy \in E(H)$. We order the vertices of G as $v_1, v_2, \ldots, v_n$ so that $x = v_{n-1}, y = v_n$, and every vertex v_i with $i < n-1$ has a neighbour v_j with $j > i$. Such an ordering exists since G is connected and H is a spanning subgraph of G. By the same argument as in the proof of Lemma 1, we can find a backbone L-colouring c of $G - \{x, y\}$ by the ordering $v_1, v_2, \ldots, v_{n-2}$. We now extend c to a backbone L-colouring c' of (G, H). Let $L'(x)$ (respectively, $L'(y)$) be the set of colours in $L(x)$ (respectively, in $L(y)$) that can be used on x (respectively, on y) to extend c. As x and y are adjacent in H, we have $|L'(x)|, |L'(y)| \geqslant 3$. Let $\alpha = \max(L'(x) \cup L'(y))$. Assign α to x if $\alpha \in L'(x)$, otherwise assign α to y; then the other vertex has at least one colour available to complete c'. □

3 Bounds in Terms of Maximum Average Degree

The proof of Lemma 1 indeed shows the following result: If the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ so that $d^+_G(v_i) + 2d^+_H(v_i) < k$, where $d^+_G(v_i)$ (or $d^+_H(v_i)$, respectively) is the number of neighbours v_j of v_i in G (in H, respectively) with $j < i$, then (G, H) is backbone k-choosable. This argument implies that any graph pair (G, H) is backbone k-choosable with $k = \lfloor \mathrm{mad}(G) + 2\mathrm{mad}(H) \rfloor + 1$. In the following result we improve this bound by 1.

Theorem 3. *Let G be a graph and H a spanning subgraph of G. Then (G, H) is k-backbone choosable, where $k = \max\{\lfloor \mathrm{mad}(G) \rfloor + 1, \lfloor \mathrm{mad}(G) + 2\mathrm{mad}(H) \rfloor\}$.*

Proof. Assume the theorem is not true. Let (G, H) be a counter example with the minimum number of vertices. For each $v \in V(G)$, define $w(v) = \max\{d_G(v) +$

$1, d_G(v) + 2d_H(v)\}$. It is obvious that the result holds if $E(H) = \emptyset$. Thus we assume that H has at least one edge. Denote $k = \lfloor \mathrm{mad}(G) + 2\mathrm{mad}(H) \rfloor$. Let L be a k-list assignment such that (G, H) is not backbone L-colourable.

Claim 1. For each $v \in V(G)$, $w(v) \geqslant k$. Furthermore, if $N_H(v) = \emptyset$, then $w(v) \geqslant k + 1$.

Proof. Suppose to the contrary, $w(v) \leqslant k - 1$. As (G, H) is a minimal counter example, and $\mathrm{mad}(G-v) \leqslant \mathrm{mad}(G)$, there is a backbone L-colouring of $(G-v, H-v)$. Assume $N_H(v) \neq \emptyset$. Since $|L(v)| = k \geqslant w(v) + 1$, there is at least one colour in $L(v)$ which extends c to a backbone L-colouring of (G, H), a contradiction.

Assume $N_H(v) = \emptyset$. Then $w(v) = d_G(v) + 1$. If $w(v) \leqslant k$, then $|L(v)| = k \geqslant d_G(v) + 1$. Thus, there is at least one colour in $L(v)$ which extends c to a backbone L-colouring of (G, H). Hence, $w(v) \geqslant k + 1$. $\square$

Let $V_k = \{v \in V(G) \mid w(v) = k\}$.

Claim 2. If $|N_H(v) \cap V_k| = q \geqslant 1$, then $w(v) \geqslant k + q + 1$.

Proof. Assume to the contrary that $w(v) \leqslant k + q$. Let $V^* = N_H(v) \cap V_k$ and $V^{**} = V^* \cup \{v\}$. As (G, H) is a minimal counter example, there exists a backbone L-colouring c of $(G - V^{**}, H - V^{**})$. For each $x \in V^{**}$, let $L'(x)$ be the set of colours in $L(x)$ which can be used on x to extend c.

For each vertex $u \in V^{**}$, the number of colours forbidden for u, due to its coloured neighbours, is at most

$$|N_G(u) \setminus V^{**}| + 2|N_H(u) \setminus V^{**}| = w(u) - d_{G[V^{**}]}(u) - 2d_{H[V^{**}]}(u), \text{ implying}$$

$$|L'(u)| \geqslant k - w(u) + d_{G[V^{**}]}(u) + 2d_{H[V^{**}]}(u).$$

Therefore, $|L'(v)| \geqslant k - k - q + q + 2q = 2q$; and for $u \in V^*$ we obtain

$$|L'(u)| \geqslant k - k + d_{G[V^{**}]}(u) + 2d_{H[V^{**}]}(u) = d_{G[V^{**}]}(u) + 2d_{H[V^{**}]}(u). \tag{1}$$

Let $G_1, G_2, \dots, G_s$ be the connected components of $G[V^*]$, and for $i = 1, 2, \dots, s$, let $H_i = G_i \cap H$. Our strategy is to find a colour that can be used on v to extend c to a backbone L-colouring c' for $(G - V^*, H - V^*)$, and then extend c' to a backbone L'-colouring of (G_i, H_i) for each i.

Note that if (G_i, H_i) is backbone degree choosable, then any extension c' of c to $(G - V^*, H - V^*)$ can be extended to a backbone L'-colouring of (G_i, H_i). In addition, if G_i has a vertex u with $|L'(u)| > d_{G[V^{**}]}(u) + 2d_{H[V^{**}]}(u)$, then by Lemma 1, again any c' can be extended to a backbone L'-colouring of (G_i, H_i). Hence, we only need to consider those components G_i such that (G_i, H_i) is not backbone degree choosable, and $|L'(u)| = d_{G[V^{**}]}(u) + 2d_{H[V^{**}]}(u)$ for all $u \in V(G_i)$. Without loss of generality, assume $G_1, G_2, \ldots, G_r$ are such components. By Theorem 2, $E(H_i) = \emptyset$ for $i = 1, 2, \ldots, r$.

For $i \in \{1, 2, \ldots, r\}$, select one vertex u_i of G_i. Define the set $L''(v)$ as follows: $L''(v) = \{j \in L'(v) : \text{for each } i = 1, 2, \ldots r,\ \{j-1, j, j+1\} \not\subseteq L'(u_i)\}$. Then

$$\begin{aligned}
|L''(v)| &\geqslant |L'(v)| - \sum_{i=1}^{r}(|L'(u_i)| - 2) \\
&\geqslant 2q - \sum_{i=1}^{r}(d_{G[V^*]}(u_i) + 3 - 2) \quad \text{(by (1))} \\
&= 2q - r - \sum_{i=1}^{r} d_{G[V^*]}(u_i) \\
&\geqslant 2q - r - (q - r) = q > 0.
\end{aligned}$$

Let c' be an extension of c to a backbone L-colouring of $(G - V^*, H - V^*)$ by assigning a colour for v from $L''(v)$. For each $x \in V^*$, let $L''(x)$ be the set of colours in $L(x)$ which can be used on x to extend c'. By (1), for each $x \in V^*$, $|L''(x)| \geqslant d_{G[V^{**}]}(x) + 2d_{H[V^{**}]}(x)$. Moreover, by the definition of $L''(v)$, we have $|L''(u_i)| \geqslant d_{G[V^{**}]}(u_i) + 2d_{H[V^{**}]}(u_i) + 1$ for each $i = 1, 2, \ldots r$. By Lemma 1, c' can be extended to a backbone L-colouring of (G, H), a contradiction. □

We use discharging method to redistribute the weight of the vertices to obtain a new weight function $w'(v)$ as follows: Each vertex v with $w(v) = k$ receives a weight of 1 from each of its neighbours in H.

Assume $w(v) = k$. By Claim 1, v has at least one neighbour in H; and by Claim 2, $w(v)$ does not lose any weight in the discharging procedure. Hence $w'(v) \geqslant k + 1$. Assume $w(v) > k$. By Claim 2, $w'(v) = w(v) - q \geqslant k + 1$, where $q = |N_H(v) \cap V_k|$. Therefore

$$\sum w(v) = \sum w'(v) \geqslant (k + 1)|V(G)|. \tag{2}$$

Let $V' = \{v \in V(G) | N_H(v) = \emptyset\}$. By the definition of maximum average degree, and noting $\mathrm{mad}(H) \geqslant 1$,

$$\begin{aligned}
\sum_{v \in V(G)} w(v) &= \sum_{v \in V(G)} (d_G(v) + \max\{1, 2d_H(v)\}) \\
&= 2|E(G)| + 4|E(H)| + |V'| \\
&\leqslant \mathrm{mad}(G)|V(G)| + 2\mathrm{mad}(H)|V(H) \setminus V'| + 2\mathrm{mad}(H)|V'| \\
&= (\mathrm{mad}(G) + 2\mathrm{mad}(H))|V(G)| \quad \text{(Because } |V(G)| = |V(H)|\text{)} \\
&< (k+1)|V(G)|.
\end{aligned}$$

This contradicts (2), and completes the proof of Theorem 3. $\square$

The bound of Theorem 3 is sharp. Let G be an odd cycle and H a spanning forest of G with $\delta(H) \geqslant 1$. By Theorem 3, $\mathrm{ch}_{\mathrm{BB}}(G, H) \leqslant \lfloor \mathrm{mad}(G) + 2\mathrm{mad}(H) \rfloor = 4$. To show $\mathrm{ch}_{\mathrm{BB}}(G, H) > 3$, we assign each vertex v of G the list $L(v) = \{1, 2, 3\}$. Then any backbone colouring of (G, H) can not use the colour 2 on any vertex v because otherwise the neighbour(s) of v in H would have no legal colour. On the other hand, since G is an odd cycle G can not be properly coloured with only two colours. Hence, $\mathrm{ch}_{\mathrm{BB}}(G, H) > 3$.

4 Bounds in Terms of the Choosability of G

Assume G is k-choosable. Let H be a subgraph of G, and let L be a list assignment of G. The proof technique to be used throughout this section is as follows. For each vertex $v \in V(G)$ construct a new list $S(v) \subseteq L(v)$ so that the following two conditions are satisfied:

Condition (A): $|S(v)| = k$.

Condition (B): If $uv \in E(H)$, then $i \pm 1 \notin S(u)$ for any $i \in S(v)$.

Once such a new list assignment S is obtained we apply the assumption that G is k-choosable to get a proper colouring of G from S; and Condition (B) will guarantee that this colouring is a backbone L-colouring of (G, H).

The *colouring number* of a graph G, denoted by $\mathrm{col}(G)$, is the least integer k such that there is an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ so that each vertex v_i has at most $k-1$ neighbours v_j with $j < i$.

Theorem 4. *Let G be a k-choosable graph and H a subgraph of G with at least one edge. Then (G, H) is backbone $k(2\mathrm{col}(H) - 1)$-choosable.*

Proof. Assume L is a list assignment of G with $|L(v)| = k(2\mathrm{col}(H) - 1)$ for all $v \in V(G)$. Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$ so that for each i, $|N_H(v_i) \cap \{v_1, v_2, \cdots, v_{i-1}\}| \leqslant \mathrm{col}(H) - 1$. We follow this ordering in assigning each vertex v_i a new list, $S(v_i) \subseteq L(v_i)$, satisfying Conditions (A) and (B).

First, select any k-subset $S(v_1) \subseteq L(v_1)$. At stage i, assume we have obtained $S_1, S_2, \cdots, S_{i-1}$ that satisfy Conditions (A) and (B). Let

$$L'(v_i) = L(v_i) \setminus \{a | \text{ for some } j < i, v_iv_j \in E(H), \text{ and } a+1 \in S(v_j) \text{ or } a-1 \in S(v_j)\}.$$

Since $|L(v_i)| = k(2\mathrm{col}(H) - 1)$ and $|N_H(v_i) \cap \{v_1, v_2, \cdots, v_{i-1}\}| \leqslant \mathrm{col}(H) - 1$, we conclude that $|L'(v_i)| \geqslant k(2\mathrm{col}(H) - 1) - 2k(\mathrm{col}(H) - 1) = k$. Let $S(v_i)$ be any k-subset of $L'(v_i)$.

It follows from the construction that the new list assignment S satisfies Conditions (A) and (B). □

Corollary 5. *Let G be a k-choosable graph. For any spanning forest T, (G, T) is backbone $3k$-choosable.*

The bound in Corollary 5 can be improved in many special cases. For a graph G, denote $\Delta(G)$ the maximum degree of a vertex in G.

Theorem 6. *Let G be a k-choosable graph. For any subgraph H of G, (G, H) is backbone $(\Delta(H) + 1)k$-choosable.*

Proof. Assume L is a list assignment of G with $|L(v)| = (\Delta(H) + 1)k$ for each vertex v.

Initially for each $v \in V(G)$, set $L'(v) = L(v)$, $n_v = 1$, and $S(v) = \emptyset$. Assume $\cup_{v \in V} L'(v) \subseteq \{1, 2, \ldots, m\}$. For $i = 1, 2, \cdots, m$, repeat the following two steps: (a) For each vertex v with $n_v = 1$ and $i \in L'(v)$, add i to $S(v)$, delete i from $L'(v)$; and for each $u \in N_H(v)$ with $i + 1 \in L'(u)$, delete $i + 1$ from $L'(u)$; (b) after completing (a), if $S(v)$ becomes a k-set for some v, set $n_v = 0$.

By following this algorithm, before $S(v)$ becomes a k-set, at most $kd_H(v)$ colours which are not added to $S(v)$ are deleted from $L'(v)$. Because $|L(v)| = (\Delta(H) + 1)k$, $S(v)$ will eventually become a k-set. Hence, Condition (A) is satisfied. It also follows easily from the algorithm that Condition (B) is satisfied. □

Corollary 7. *If G is k-choosable, and M is a matching in G, then (G, M) is backbone $2k$-choosable.*

Corollary 8. *If G is k-choosable, and H is a disjoint union of cycles and paths, then (G, H) is backbone $3k$-choosable.*

We say a graph is *unicyclic* if it contains at most one cycle.

Corollary 9. *If G is k-choosable, and each component of H is unicyclic, then (G, H) is backbone $3k$-choosable.*

Proof. Assume L is a list assignment of G with $|L(v)| = 3k$ for all vertices v. For each component H_i of H, let C_i be the set of vertices of the cycle in H_i; if H_i is acyclic, let C_i be a singleton set of an arbitrary vertex of H_i. Use the algorithm presented in the proof of Theorem 6 to obtain a new list $S(v)$ of k colours for each $v \in C_i$.

For the vertices in H which have not yet been assigned a new list, we proceed as follows. If u has not been assigned a new list and is adjacent, in H, to a vertex v which has been assigned a new list, take $S(u)$ to be any k-subset of $L(u) \setminus \{j | j + 1 \in S(v) \text{ or } j - 1 \in S(v)\}$. As each component of H contains at most one cycle, each u has only one such neighbour. In addition, because $|L(u)| = 3k$, so $L(u) \setminus \{j | j + 1 \in S(v) \text{ or } j - 1 \in S(v)\}$ has size at least k, hence the k-subset $S(u)$ exists, satisfying Condition (A).

Finally, let $S(v)$ be any k-subset of $L(v)$ for every remaining vertex v in G. It is easy to check that the new list assignment S of G satisfies Conditions (A) and (B). $\square$

By considering a special family of subgraphs H for Corollary 8, the bound can be improved as follows.

Theorem 10. *If G is k-choosable, and each component of H is a path of length at most 2, then (G, H) is backbone $(2k + 1)$-choosable.*

Proof. Assume L is a list assignment of G with $|L(v)| = 2k + 1$ for all vertices v of G. If $d_H(v) = 0$, let $S(v)$ be any k-subset of $L(v)$. If (x_1, x_2) is a path of length 1 which is a component of H, then we choose $S(x_1)$ and $S(x_2)$ as in the proof of Theorem 6.

It remains to consider those components of H which are paths of length 2. Let (x_1, y, x_2) be such a component of H, a path of length 2. A triple (Y, X_1, X_2) is *good* if all the following hold:

(i) $Y \subseteq L(y)$, $|Y| \leqslant k$.

(ii) $|X_1| = |X_2| \leqslant k$, and $X_i \subseteq L(x_i) \setminus \{j \pm 1 : j \in Y\}$ for $i = 1, 2$.

(iii) $|L(x_i) \setminus \{j \pm 1 : j \in Y\}| \geqslant |L(x_i)| - |Y|$ for $i = 1, 2$.
Equivalently, $|L(x_i) \cap \{j \pm 1 : j \in Y\}| \leqslant |Y|$ for $i = 1, 2$.

(iv) $|L(y) \setminus \{j \pm 1 : j \in X_1 \cup X_2\}| \geqslant |L(y)| - |X_1|$.
Equivalently, $|L(y) \cap \{j \pm 1 : j \in X_1 \cup X_2\}| \leqslant |X_1|$.

Note that $(\emptyset, \emptyset, \emptyset)$ is a trivial good triple. We choose a good triple (Y, X_1, X_2) so that $|Y| + |X_1| + |X_2|$ is maximized. If $|Y| = k$, then let $S(y) = Y$ and let $S(x_i)$ be any k-subset of $L(x_i) \setminus \{i \pm 1 | i \in Y\}$. We are done, as the new list satisfies Conditions (A) and (B). If $|X_1| = k$, then let $S(x_i) = X_i$ and let $S(y)$ be any k-subset of $L(y) \setminus \{i \pm 1 | i \in X_1 \cup X_2\}$, we are done again.

Thus we assume that $|Y| < k$ and $|X_i| < k$. Let

$$L'(x_i) = L(x_i) \setminus (X_i \cup \{i \pm 1 \mid i \in Y\});$$

$$L'(y) = L(y) \setminus (Y \cup \{i \pm 1 \mid i \in X_1 \cup X_2\}).$$

By the maximality of (Y, X_1, X_2), we have

(1) If $c \in L'(y)$, then there is a $j \in \{1, 2\}$ such that $\{c \pm 1\} \subseteq L'(x_j)$. For otherwise, $(Y \cup \{c\}, X_1, X_2)$ is a good triple.

(2) If $c \in L'(x_1) \cap L'(x_2)$, then $\{c \pm 1\} \subseteq L'(y)$. For otherwise, $(Y, X_1 \cup \{c\}, X_2 \cup \{c\})$ is a good triple.

Let $c = \min L'(y)$. By (1) and (2), $c - 1 \in L'(x_1) \cup L'(x_2)$ and $c - 1 \notin L'(x_1) \cap L'(x_2)$. Without loss of generality, we assume $c - 1 \in L'(x_1) \setminus L'(x_2)$. Let l be the smallest nonnegative integer such that $c + 2l + 1 \notin L'(x_1) \cap L'(x_2)$.

For two integers a and b of the same parity, we denote $[a, b]_2$ the set of integers $\{a, a + 2, a + 4, \cdots, b\}$.
Claim. $[c, c + 2l]_2 \subseteq L'(y)$ and $|Y \cup [c, c + 2l]_2| < k$.

Proof. For the first part, if $l = 0$, then as $c \in L'(y)$, we are done. Assume $l \geqslant 1$. Then $[c + 1, c + 2l - 1]_2 \subseteq L'(x_1) \cap L'(x_2)$. By (2) repeatedly, we have $[c, c + 2l]_2 \subseteq L'(y)$.

Now we prove the second part. Assume to the contrary, there exists $l' \leqslant l$ such that $|Y \cup [c, c+2l']_2| = |Y| + (l'+1) = k$. By (iii), $|L(x_i) \setminus \{j \pm 1 : j \in Y\}| \geqslant |L(x_i)| - |Y| = 2k+1-|Y|$. As $|L(x_i) \cap \{j \pm 1 : j \in [c, c+2l']_2| \leqslant l'+2$, we have $|L(x_i) \setminus \{j \pm 1 : j \in Y \cup [c, c+2l']_2| \geqslant 2k+1-|Y|-(l'+2) = k$.

Let $S(y) = Y \cup [c, c+2l']_2$ and $S(x_i)$ be any k-subsets of $L(x_i) \setminus \{j \pm 1 : j \in Y \cup [c, c+2l']_2$, we are done. This completes the proof of Claim. $\square$

By (1), $c+2l+1 \in L'(x_1) \cup L'(x_2)$. If $c+2l+1 \in L'(x_2) \setminus L'(x_1)$, then $(Y \cup [c, c+2l]_2, X_1, X_2)$ is a good triple, contrary to our assumption. Thus we have $c+2l+1 \in L'(x_1) \setminus L'(x_2)$.

If $|L(x_1) \setminus \{j \pm 1 : j \in Y\}| > |L(x_1)| - |Y|$, then by (iii), $(Y \cup [c, c+2l]_2, X_1, X_2)$ is again good, a contradiction. Thus, $|L(x_1) \setminus \{j \pm 1 : j \in Y\}| = |L(x_1)| - |Y|$ (or equivalently, $|L(x_1) \cap \{j \pm 1 : j \in Y\}| = |Y|$), which implies that $|L'(x_2)| \geqslant |L'(x_1)|$. As $c-1 \in L'(x_1) \setminus L'(x_2)$, we get $L'(x_2) \setminus L'(x_1) \neq \emptyset$.

Let $d = \min(L'(x_2) \setminus L'(x_1))$. We show that $d-1 \notin L'(y)$ and $d+1 \in L'(y)$. First, assume $d-1 \in L'(y)$. By (1), $d-2 \in L'(x_2)$. By the minimality of d, $d-2 \in L'(x_1) \cap L'(x_2)$. Then by (2), $d-3 \in L'(y)$. Repeating this argument, we have $d-1-2j \in L'(y)$ and $d-2-2j \in L'(x_1) \cap L'(x_2)$ for all j, which is an obvious contradiction (note, if $d-2-2j \in L'(x_1) \setminus L'(x_2)$ for some j, then we again get a contradiction with a larger good triple). Hence $d-1 \notin L'(y)$. Secondly, assume $d+1 \notin L'(y)$. Then $(Y, X_1 \cup \{c-1\}, X_2 \cup \{d\})$ is good, a contradiction. So, $d+1 \in L'(y)$.

Let m be the smallest positive integer such that $d+2m \notin L'(x_1) \cap L'(x_2)$. By the same argument as in the proof of Claim, we can show that $[d+1, d+2m-1]_2 \subseteq L'(y)$, and $|Y \cup [d+1, d+2m-1]_2| < k$. If $d+2m \notin L'(x_2)$, then $(Y \cup [d+1, d+2m-1]_2, X_1, X_2)$ is good, contrary to the maximality of (Y, X_1, X_2). Thus, $d+2m \in L'(x_2) \setminus L'(x_1)$. If $l+m+1 \leqslant k-|Y|$, then $(Y \cup [c, c+2l]_2 \cup [d+1, d+2m-1]_2, X_1, X_2)$ is good, a contradiction. If $l+m+1 > k-|Y|$, then let $S(y) = Y \cup [c, c+2l]_2 \cup [d+1, d+2(k-|Y|-l)-3]_2$; and for $j = 1, 2$, let $S(x_j) = L(x_j) \setminus \{c \pm 1 | c \in S(y)\}$. It is straightforward to verify that the new list assignment S of G satisfies Conditions (A) and (B). $\square$

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