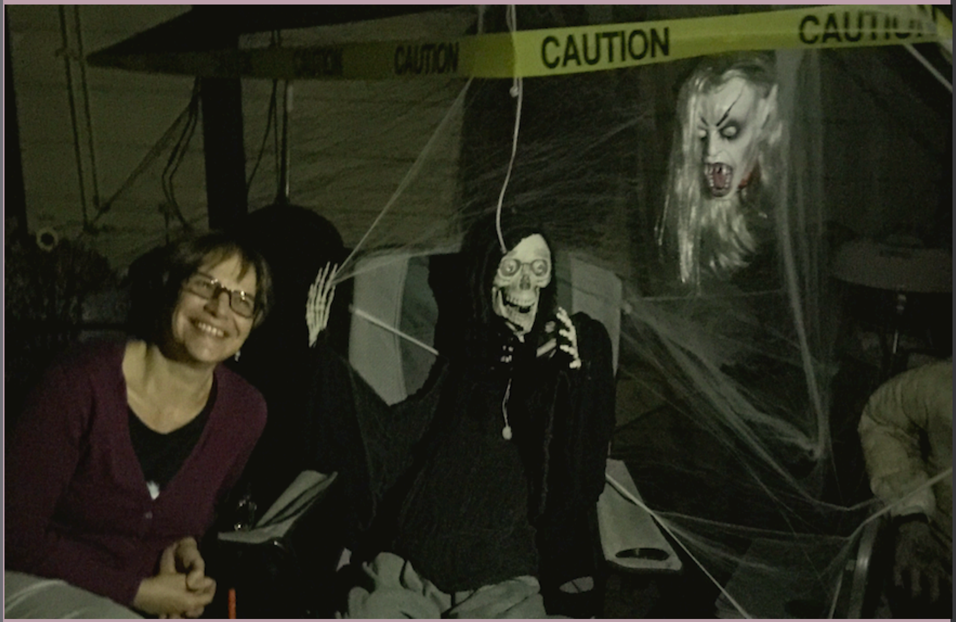


Logarithms: Trick or Treat?

Physics Colloquium
Cal State LA
October 31, 2019

The time is 1594. The place is Scotland.

You are visiting with John Napier, 8th Laird of Merchiston.

He is a Scottish landowner known as a mathematician, physicist, and astronomer

How did Napier do Math?

Physics and astronomy computations involved sine and cosine functions

Useful tool:

$$\cos a \cos b = \frac{\cos(a-b) + \cos(a+b)}{2}$$

Multiplication is replaced by addition!

Problem: Find 105×720.

1. Scale values to interval [-1,1]: 0.105 and 0.720
2. Find angles whose cosines are close to those values: $\cos 84° \approx 0.105$ and $\cos 44° \approx 0.720$
3. Calculate sum and difference of these angles:

$$84° - 44° = 40° \text{ and } 84° + 44° = 128°$$

4. Find cosine values and average them:

$$\cos 40° \approx -0.616 \text{ and } \cos 128° \approx 0.766$$

$$\frac{1}{2}(\cos 40° + \cos 128°) \approx 0.075$$

5. Scale back by shifting 6 decimal places to the right to obtain 75,000
6. Actual answer is 75,600

How did Napier do Math?

Arithmetic and geometric progressions

Arithmetic	0	1	2	3	4	5
geometric	1	10	100	1,000	10,000	10,0000
Arithmetic	0	1	2	3	4	5
geometric	1	2	4	8	16	32
Arithmetic	0	1	2	3	4	5
geometric	1	1.1	1.21	1.331	1.4641	1.61051

Observations:

- Multiplying numbers with the same base reduces to addition
- Dividing numbers with the same base reduces to subtraction
- A base close to 1 makes for a list of values with smaller gaps between them

Naperian Logarithms

- Napier choose $0.9999999 = 1 - 10^{-7}$ as the base
- Had factor of 10^7 to create integers for his computations

$$\text{NapLog}(N) \;=\; L \text{ if } \quad \boldsymbol{N = 10^7(0.9999999)^L}$$

$$\text{NapLog}(\sqrt{N_1 N_2}) \;=\; \frac{1}{2}(\text{NapLog}\, N_1 \;+ \text{NapLog}\, N_2)$$
$$\text{NapLog}(10^{-7} N_1 N_2) \;=\; \text{NapLog}\, N_1 \;+ \text{NapLog}\, N_2$$
$$\text{NapLog}\left(10^7 \frac{N_1}{N_2}\right) \;=\; \text{NapLog}\, N_1 - \text{NapLog}\, N_2$$

- He labored for a total of 20 years to develop the idea and to create tables

LOGARITHMORVM CANONIS DESCRIPTIO,

SEV

ARITHMETICARVM SVPPVTATIONVM MIRABILIS ABBREVIATIO.

Eiusque vsus in vtraque Trigonometria *vt etiam in omni* Logistica Mathematica, *amplissimi, facillimi & expeditissimi explicatio.*

Authore ac Inuentore IOANNE NEPERO, Barone Merchistonij, &c. SCOTO.

LVGDVNI,

Apud Barth. Vincentium.

M. DC. XX.

Cum Priuilegio Cæsar. Majest. & Christ. Galliarum Regis.

Gr. 30

30

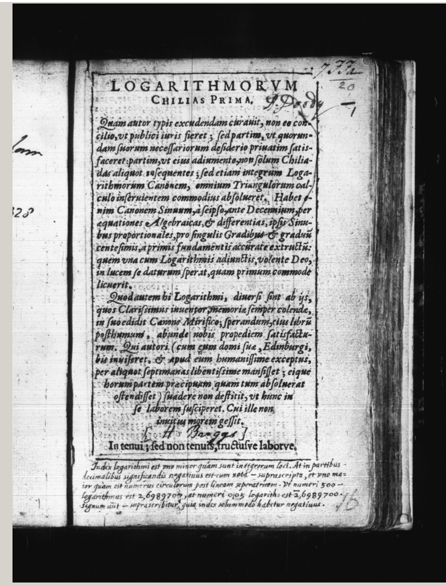
min	Sinus	Logarithmi	Differentiæ	logarithmi	Sinus
0	5000000	6931469	5493059	1438410	8660254
1	5002519	6926432	5486342	1440090	8658799
2	5005038	6921399	5479628	1441771	8657344
3	5007556	6916369	5472916	1443453	8655888
4	5010074	6911342	5466206	1445136	8654431
5	5012591	6906319	5459498	1446821	8652973
6	5015108	6901299	5452792	1448507	8651514
7	5017624	6896282	5446088	1450194	8650055
8	5020140	6891269	5439387	1451882	8648595
9	5022656	6886259	5432688	1453571	8647134
10	5025171	6881253	5425992	1455261	8645673
11	5027686	6876250	5419298	1456952	8644211
12	5030200	6871250	5412605	1458645	8642748
13	5032714	6866254	5405915	1460339	3641284
14	5035227	6861261	5399227	1462034	8639820
15	5037740	6856271	5392541	1463730	8638355
16	5040253	6851285	5385858	1465427	8636889
17	5042765	6846302	5379177	1467125	8635423
18	5045277	6841323	5372499	1468824	8633956
19	5047788	6836347	5365822	1470525	8632488
20	5050299	6831374	5359147	1472227	8631019
21	5052809	6826405	5352475	1473930	8629549
22	5055319	6821439	5345805	1475634	8628079
23	5057829	6816476	5339137	1477339	8626608
24	5060338	6811516	5332471	1479045	8625137
25	5062847	6806560	5325808	1480752	8623665
26	5065355	6801607	5319147	1482460	8622192
27	5067863	6796657	5312488	1484169	8620718
28	5070370	6791710	5305831	1485879	8619243
29	5072877	6786767	5299177	1487590	8617768
30	5075384	6781827	5292525	1489302	8616292

59

Mirifici Logarithmorum Canonis Descriptio, 1614

Common Logarithms

- Both in 1615 and 1616, Henry Briggs, an English Mathematician, visited with Napier to discuss his new invention
- Napier and Briggs agreed on improvements: base 10 and 0 = Log(1).
- In 1617, Briggs published *Logarithmorum Chilias Prima,* which contained the logarithms to base 10 of numbers from 1 to 1,000, calculated to 14 decimal places.
- In 1624, Briggs published the *Arithmetica Logarithmica*, which contained tables of logarithms from 1 to 20,000 and from 90,001 to 100,000, calculated to 14 decimal places.

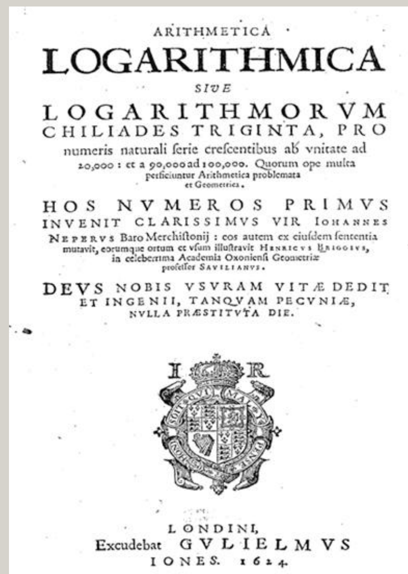


LOGARITHMORVM
CHILIAS PRIMA,

Quam autor typis excudendam curauit, non eo concilio, vt publici iuris fieret; sed partim, vt quorundam suorum necessariorum desiderio priuatim satisfaceret: partim, vt eius adiumento, non solum Chiliadas aliquot sequentes; sed etiam integrum Logarithmorum Canonem, omnium Triangulorum calculo inseruientem commodius absolueret. Habet enim Canonem Sinuum, à seipso, ante Decennium, per æquationes Algebraicas, & differentias, ipsis Sinubus proportionales, pro singulis Gradibus & graduũ centesimis, à primis fundamentis accuratè extructũ: quem vna cum Logarithmis adiunctis, volente Deo, in lucem se daturum sperat, quam primum commodè licuerit.

Quod autem hi Logarithmi, diuersi sint ab iis, quos Clarissimus inuentor, memoriæ semper colendæ, in suo edidit Canone Mirifico; sperandum, eius librũ posthumum, abunde nobis propediem satisfacturum. Qui autori (cum eum domi suæ, Edinburgi, bis inuiseret, & apud eum humanissime exceptus, per aliquot septimanas libentissime mansisset; eique horum partem præcipuam, quam tum absoluerat ostendisset) suadere non destitit, vt hunc in se laborem susciperet. Cui ille non inuitus morem gessit.

In tenui; sed non tenuis, fructusve laborve.



ARITHMETICA
LOGARITHMICA
SIVE
LOGARITHMORVM
CHILIADES TRIGINTA, PRO
numeris naturali serie crescentibus ab vnitate ad 20,000: et a 90,000 ad 100,000. Quorum ope multa perficiuntur Arithmetica problemata et Geometrica.

HOS NVMEROS PRIMVS INVENIT CLARISSIMVS VIR IOHANNES NEPERVS Baro Merchistonij: eos autem ex eiusdem sententia mutavit, eorumque ortum et vsum illustravit HENRICVS BRIGGIVS, in celeberrima Academia Oxoniensi Geometriæ professor SAVILIANVS.

DEVS NOBIS VSVRAM VITÆ DEDIT ET INGENII, TANQVAM PECVNIÆ, NVLLA PRÆSTITVTA DIE.

LONDINI,
Excudebat GVLIELMVS IONES. 1624.

The Slide Rule

5 March 1574 – 30 June 1660

Invented in 1622 by by William Ougthred

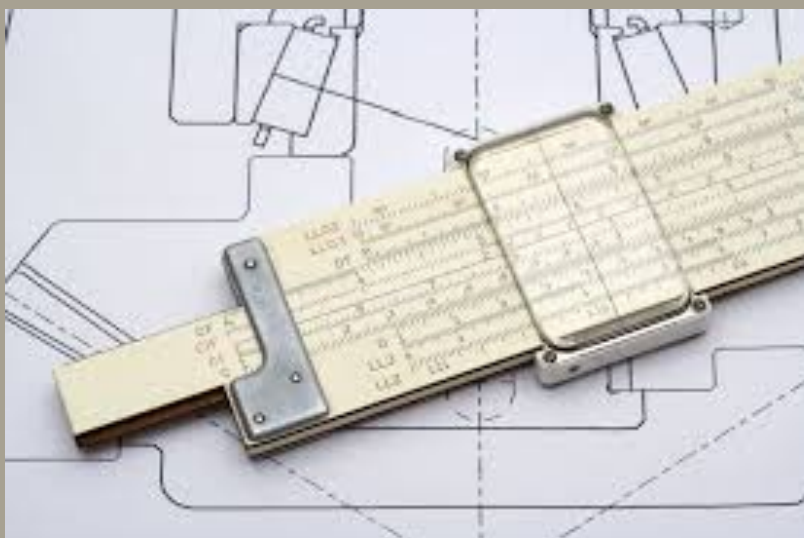
Modern version invented by Amédée Mannheim in 1859

17 July 1831 – 11 Dec 1906

Before the electronic calculator, the most commonly used calculation tool in science and engineering.

Primarily for multiplication, division, powers, roots, and trigonometry, with specialized versions for aviation, finance,…

Became obsolete around 1974 with the introduction of the handheld electronic scientific calculator.



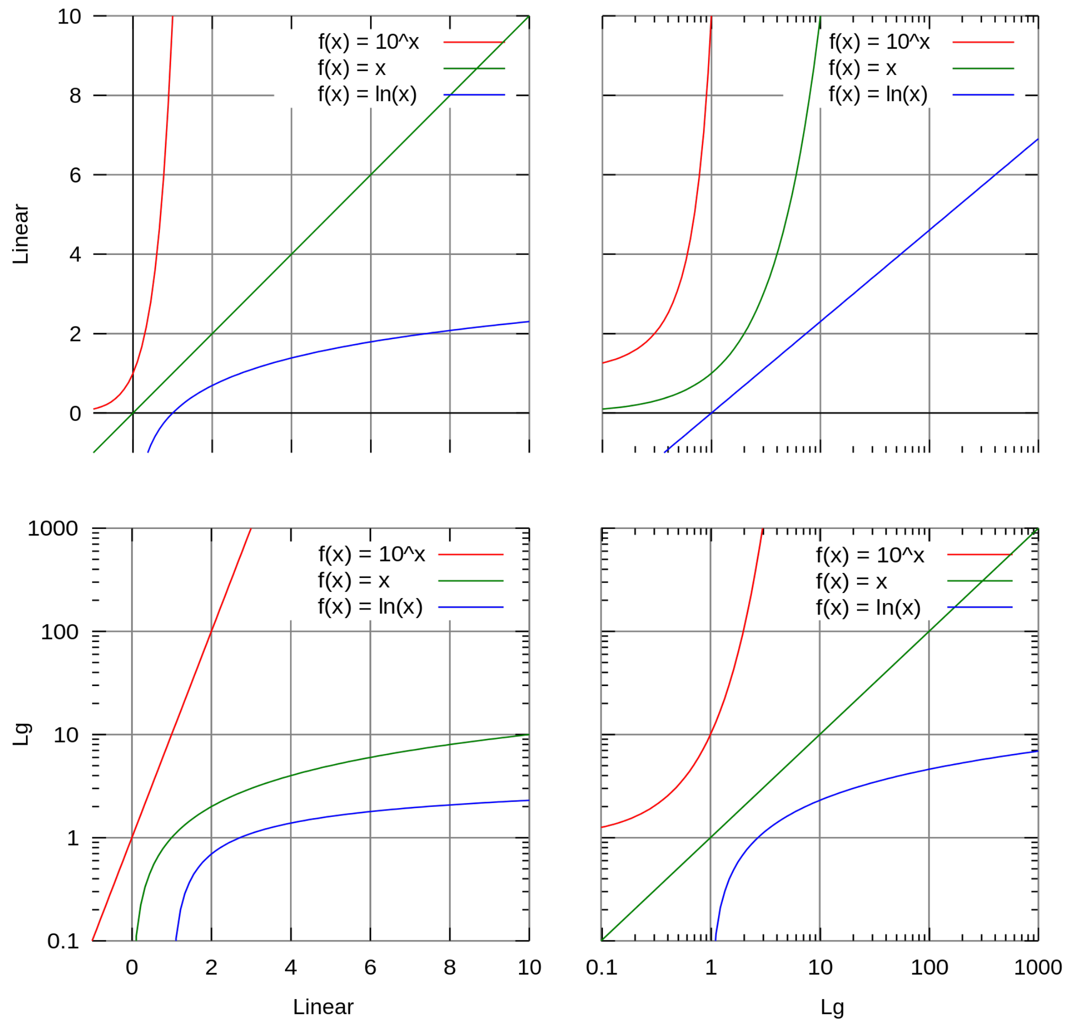
Pull out your slide rules

ihoe / Getty Images

Log Scale versus Linear Scale

Where on this scale is 1,000?

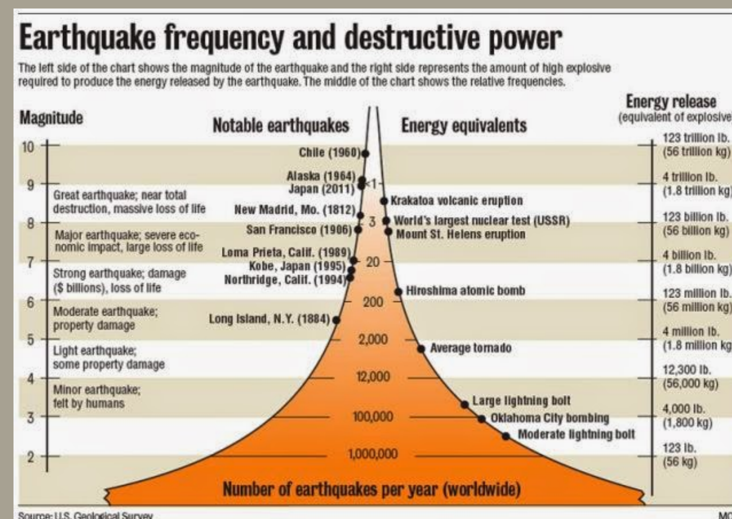
Linear scale does not work so well when we have data that is very different in (multiplicative) scale.



Logarithmic scale versus linear scale

Applications in STEM disciplines

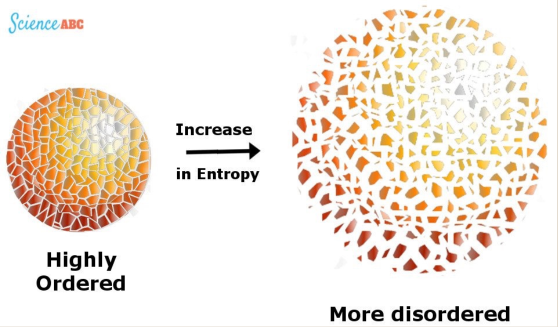
Richter magnitude scale and moment magnitude scale (MMS) for strength of earthquakes and movement in the earth



Applications in STEM disciplines

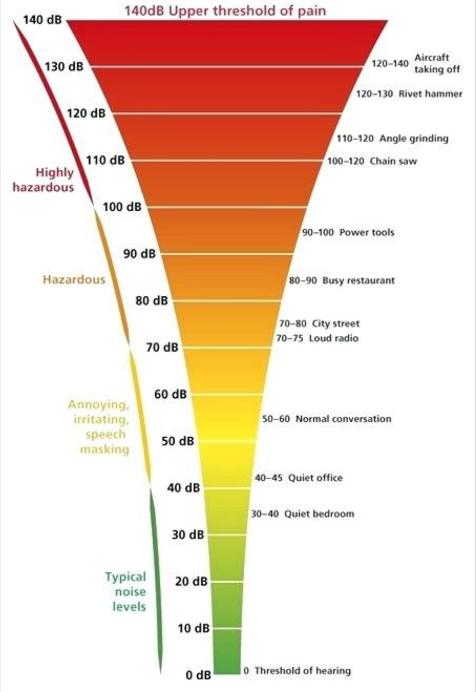
Entropy in Thermodynamics

$S = k_B \ln \Omega$



Sound intensity

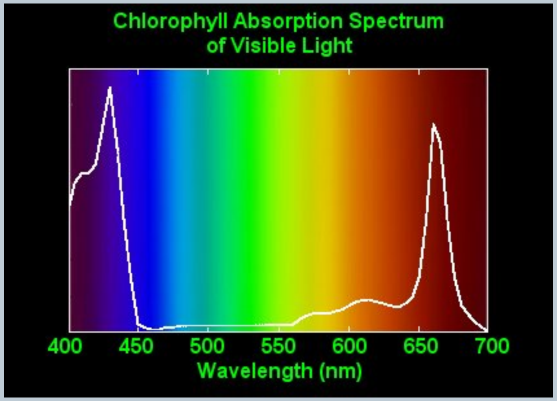
$I = 10\log_{10}\left(\frac{I}{I_0}\right)$



Applications in STEM disciplines

Stellar magnitude for brightness

$$m - m_r = -2.5 \log_{10}\left(\frac{I_1}{I_r}\right)$$



Absorbance of light

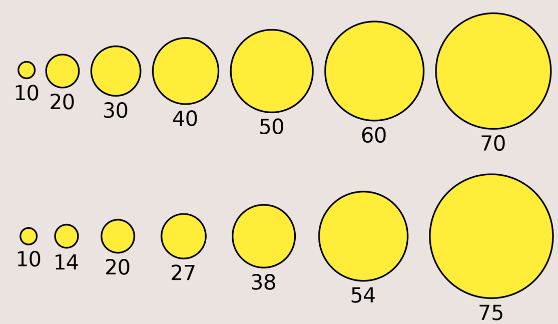
$$A = \log_{10}\left(\frac{\Phi_e^i}{\Phi_e^t}\right)$$

and more .....

Applications in other disciplines

Frequency level for the relative pitch of notes in music scale

Counting f-stops for ratios of photographic exposure



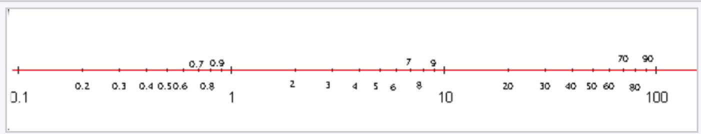
Some of our senses operate in a logarithmic fashion (Weber–Fechner law)

Benford's Law

- In 1881, Newcomb observed that in tables of logarithms the first pages are much more worn than later pages
- In 1938, Benford investigated a variety of real data and listed the distributions of the first digits.
- Observations closely followed this "law"

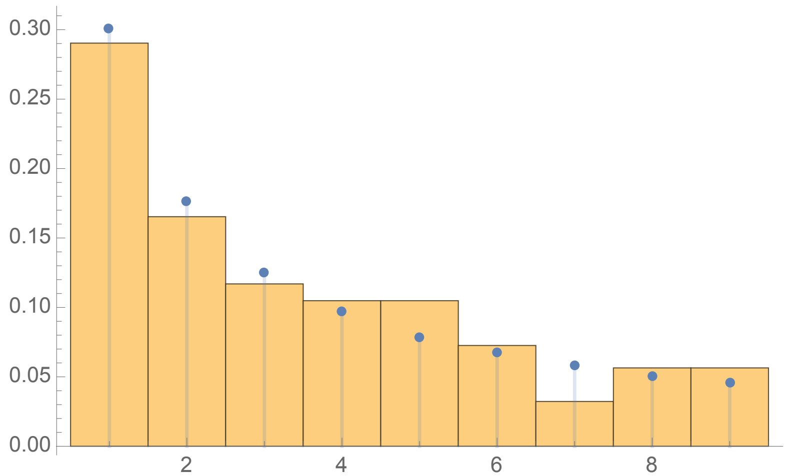
$$P(d) = \log_{10}(d+1) - \log_{10}(d)$$

$$= \log_{10}\left(\frac{d+1}{d}\right)$$

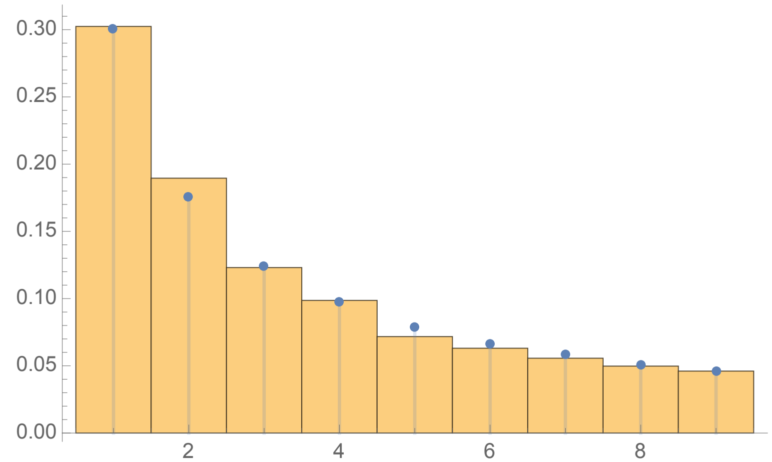


A logarithmic scale bar. Picking a random *x* position uniformly on this number line, roughly 30% of the time the first digit of the number will be 1.

- **TV trivia**: Benford's law was used by the character Charlie Eppes as an analogy to help solve a series of high burglaries in Season 2 of *NUMB3RS*.



249 world countries



3234 US counties

Benford's Law for first digits of populations


Thank you!
Any
Questions????

References used and for further study

- https://en.wikipedia.org/wiki/John_Napier
- https://www.britannica.com/biography/John-Napier#ref241319
- https://www.youtube.com/watch?v=dT7bSn03lx0 (training video for slide rule operation)
- https://en.wikipedia.org/wiki/Slide_rule
- https://en.wikipedia.org/wiki/Prosthaphaeresis
- http://mathworld.wolfram.com/BenfordsLaw.html
- https://archive.org/details/constructionofwo00napiuoft/page/n8 (English translation of Napier's second book)
- https://en.m.wikipedia.org/wiki/Logarithmic_scale
- Newcomb, S. "Note on the Frequency of the Use of Digits in Natural Numbers." *Amer. J. Math.* **4**, 39-40, 1881
- Hill, T. P. "The First Digit Phenomenon." *Amer. Sci.* **86**, 358-363, 1998.
- Benford, F. "The Law of Anomalous Numbers." *Proc. Amer. Phil. Soc.* **78**, 551-572, 1938.
- Eli Maor, e: The story of a number

Photo Credit

- https://www.thoughtco.com/history-of-the-slide-rule-1992408
- https://www.sliderulemuseum.com/
- http://endgameviable.com/a-year-of-blog-stats/500004176-03-01/
- https://en.wikipedia.org/wiki/Slide_rule
- https://www.scienceabc.com/wp-content/uploads/2016/11/Entropy.jpg
- http://www.pmonta.com/tables/logarithmorum-chilias-prima/index.html

Log-Lin Scale for Exponential phenomena

$y = y_0 b^x$ Take logs on both sides

$\ln y = \ln y_0 + x \ln b$ Set $c_1 = \ln y_0$, $c_2 = \ln b$, $z = \ln y$

$z = c_1 + c_2 \cdot x$ $\ln y$ is a **linear** function of x

Can use linear regression on the transformed data to estimate y_0 and b.