Weds
2/12
week 4

## Previously

**Thm 29 / Thm 37 (Euclid)**

Let $n \geq 2$.
If $2^n - 1$ is prime,
then $2^{n-1}(2^n - 1)$
is an even perfect
number.

N713

Thm 38 Let $a, b, c$ be positive integers.
If $a \mid bc$ and $\gcd(a,b)=1$, then $a \mid c$.

pf: Math 4460. $\blacksquare$

Ex 39: $6 \mid 120$

$6 \mid 5 \cdot 24 \xrightarrow{\gcd(6,5)=1} 6 \mid 24$

$a \mid b\,c$

Lemma 40: If $\ell \geq 1$ and $t$ is an odd positive integer, then $\gcd(2^\ell, t) = 1$.

Proof:
The positive divisors of $2^\ell$ are
$$1, 2, 2^2, \ldots, 2^\ell.$$
Since $t$ is odd the only positive common divisor of $2^\ell$ and $t$ is 1.

So, $\gcd(2^\ell, t) = 1$. $\blacksquare$

Thm 4.1 (Euler)

Any even perfect number must be of the form

$$2^{n-1}(2^n - 1)$$

where $2^n - 1$ is prime and $n \geq 2$.

proof: Suppose $x$ is an even perfect number.

Since $x$ is even, $\boxed{x = 2^{n-1}\cdot m}$ where $n \geq 2$ and $m$ is a positive odd integer.

Since $x$ is perfect we have $\boxed{\sigma(x) = 2x = 2^n m}$ (*)

By lemma 40, we have $\gcd(2^{n-1}, m) = 1$.

Thus,

$$\sigma(x) = \sigma(2^{n-1}\cdot m) = \sigma(2^{n-1})\,\sigma(m) = \left(\frac{2^n - 1}{2-1}\right)\sigma(m) = (2^n - 1)\,\sigma(m).$$

$\boxed{\text{If } \gcd(a,b)=1, \text{ then } \sigma(ab) = \sigma(a)\sigma(b)}$

$\sigma(p^e) = 1 + p + p^2 + \cdots + p^e = \dfrac{p^{e+1} - 1}{p - 1}$ if $p$ is prime

**Side example**

$x = 28$

$28 = 2^2 \cdot 7 = 2^{3-1}\cdot 7$

$2^{n-1}\cdot m$

So, $\boxed{\sigma(x) = (2^n - 1)\,\sigma(m)}$ $(**)$

By $(*)$ and $(**)$ we get

$$2^n m = (2^n - 1)\,\sigma(m).$$

So, $(2^n - 1) \mid 2^n \cdot m.$

Since $(2^n - 1) \mid 2^n \cdot m$

and $\gcd(\underbrace{2^n-1}_{\text{odd}}, 2^n) = 1$,

By lemma 40

by Thm 38 we have $(2^n - 1) \mid m$

So, $m = (2^n - 1)M$

where $M$ is a positive integer and $M < m$

Substi

into

gives

$2^n$

Thus,

Substituting $m=(2^n-1)M$

into $2^n m = (2^n-1)\sigma(m)$

gives

$$2^n(2^n-1)M = (2^n-1)\sigma(m).$$

Thus, $2^n M = \sigma(m)$.

nd $M<m$

Using this equation and the fact that $m$ and $M$ are divisors of $m$ we get

$$2^n M = \sigma(m) \geq m + M = (2^n-1)M + M = 2^n M$$

So, $\boxed{2^n M} = \sigma(m) \geq m + M = \boxed{2^n M}$.

You can't have $>$ above since that would imply $2^n M > 2^n M$.

M

So,

$$\sigma(m) = m + M \quad (*)$$

Since $m$ & $M$ are divisors of $m$, this implies the only divisors of $m$ are $m$ & $M$.

So, $M = 1$.

So, $m$ is prime since its only divisors are $m$ & $1$.

And

$$m = (2^n - 1)M \underset{\boxed{M=1}}{=} 2^n - 1.$$

Thus,

$$x = 2^{n-1} m = 2^{n-1}(2^n - 1)$$

where $2^n - 1$ is prime & $n \geq 2$. $\blacksquare$

## Summary

**Thm 42** (Euclid / Euler)

$x$ is an even perfect number

if and only if

$x = 2^{n-1}(2^n - 1)$ where

$n \geq 2$ and $2^n - 1$ is prime.

So there is a correspondence between even perfect numbers and primes of the form $2^n - 1$.

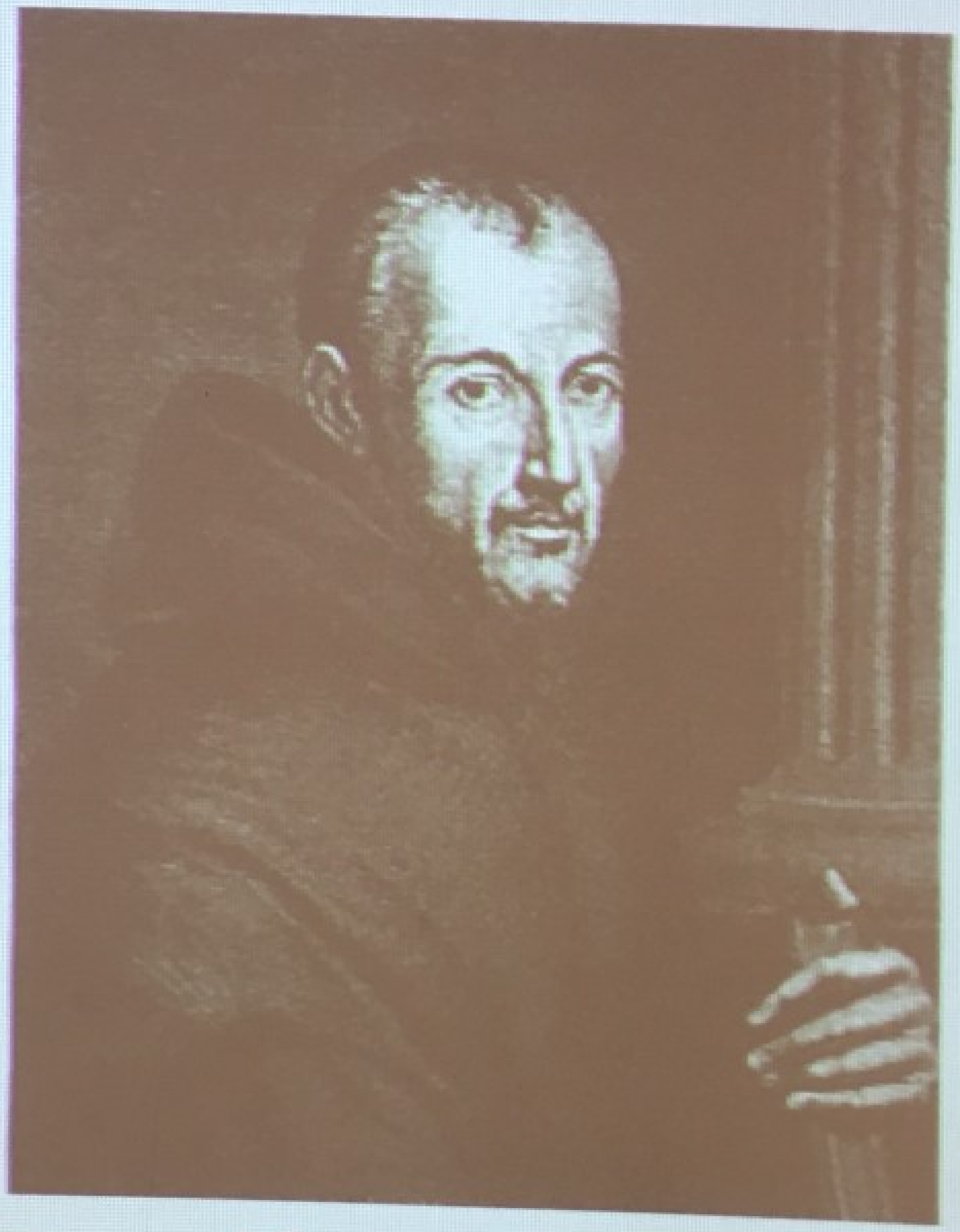
Def 43: Let $n$ be a positive integer. Then

$M_n = 2^n - 1$ is called a Mersenne number.

If $M_n = 2^n - 1$ is prime, then its called a Mersenne prime

[mɛʀsɛn]; 8
ose works
nong
written in the
s laws, which
on guitars
rselle, for
ordained
"the center
600s"[3] and,
as, "the post-
r and wrote

**Marin Mersenne**



| | |
|---|---|
| **Born** | 8 September 1588<br>Oizé, Maine, France |
| **Died** | 1 September 1648 (aged 59)<br>Paris, France |
| **Nationality** | French |
| **Known for** | Mersenne primes<br>Mersenne's laws |

**Scientific career**

| | |
|---|---|
| **Influences** | René Descartes<br>Étienne Pascal<br>Pierre Petit<br>Gilles de Roberval |

# Mersenne numbers

$M_1 = 2^1 - 1 = 1$ not prime

$M_2 = 2^2 - 1 = 3$ Mersenne Prime

$M_3 = 2^3 - 1 = 7$

$M_4 = 2^4 - 1 = 15$ not prime

$M_5 = 2^5 - 1 = 31$

$M_6 = 2^6 - 1 = 63 \leftarrow$ not prime

$M_7 = 2^7 - 1 = 127 \leftarrow$ Mersenne prime

$\left.\begin{matrix} M_8 \\ M_9 \\ M_{10} \end{matrix}\right\}$ not prime

$M_{11} = 2^{11} - 1 = 2047 = (28)(89)$ not prime

Thm 44

If $M_n = 2^n - 1$ is prime, then $n$ is prime.

The converse: "If $n$ is prime then $M_n$ is prime" is not true. $\boxed{M_{11} \text{ is not prime}}$

proof: Suppose $M_n = 2^n - 1$ is prime. We show $n$ is prime.

What if $n$ is not prime?

Then $n = rs$ with $1 < r$, $1 < s$.

So,

$$(*)\quad 2^n - 1 = 2^{rs} - 1 = \underbrace{(2^r - 1)}_{>1}\underbrace{(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 2^r + 1)}_{>1}$$

Since $r > 1$,

$$2^r - 1 > 1.$$

Since $s > 1$,

$$2^{r(s-1)} + 2^{r(s-2)} + \dots + 2^r + 1 \overset{\boxed{s>1}}{\geq} 2^r + 1 \overset{\boxed{\substack{r>1 \\ r\geq 2}}}{\geq} 5 > 1$$

So, (*) shows that $2^n - 1$ is not prime.

This contradicts our assumption that $2^n - 1$ is prime.

So, $n$ is prime. ∎