Monday
2/17

Previously

Euclid/Euler

X is an even perfect number iff

$$X = 2^{n-1}(2^n - 1)$$

where $2^n - 1$ is prime, $n \geq 2$.

$M_n = 2^n - 1$ is called a Mersenne number

If $M_n = 2^n - 1$ is prime, then n is prime.

Converse: "If n is prime, then $M_n = 2^n - 1$ is prime" is not true. $n = 11$ is prime but $M_{11} = 2^{11} - 1 = 2047 = (23)(89)$ is not prime

New stuff

First few perfect numbers

6
28
496
8128
33,550,336
8,589,869,056
137,438,691,328
⋮

they all end in 6 or 8

Ex 45:

$$496 = 49(10) + 6$$
$$\equiv 0 + 6 \pmod{10}$$
$$\equiv 6 \pmod{10}.$$

Idea: modding by 10 picks out the last digit.

Lemma 46:

$16^t \equiv 6 \pmod{10}$ for $t \geq 1$

pf by induction:

When $t = 1$,

$$16^1 \equiv 10 + 6 \equiv 0 + 6 \equiv 6 \pmod{10}$$

Let $t \geq 1$ and suppose $16^t \equiv 6 \pmod{10}$.

Then,

$$\begin{aligned}
16^{t+1} &\equiv (16^t)(16) \pmod{10} \\
&\equiv (6)(16) \pmod{10} \\
&\equiv (6)(6) \pmod{10} \\
&\equiv 36 \pmod{10} \equiv 6 \pmod{10}.
\end{aligned}$$

$\boxed{16 \equiv 6}$

So, by induction $16^t \equiv 6 \pmod{10}$ for all $t \geq 1$. $\blacksquare$

Fact 47:

Let k be an odd integer.
If we divide 4 into k
we get $k = 4q + r$
with $0 \le r < 4$.

So,

$k = 4q$ ← even
or $k = 4q+1$ ← odd
or $k = 4q+2$ ← even
or $k = 4q+3$ ← odd

≥ 1,

$\blacksquare$

So, if k is an odd integer then
$k = 4q+1$ or
$k = 4q+3$
where $q \in \mathbb{Z}$.

Thm 48:

If x is an even perfect number, then x ends in a 6 or an 8.

proof: Since x is an even perfect number

$$x = 2^{n-1}(2^n - 1)$$

where 2^n-1 is prime and $n \geq 1$.

We also know that n is prime.

case 1: $n=2$

$$x = 2^{2-1}(2^2-1) = 2 \cdot 3 = 6$$

which ends in 6.

2 is the only even prime so we can assume n is odd.

Since n is odd either

$n = 4q+1$ or $n = 4q+3$ where $q \in \mathbb{Z}$,

case 2: $n=4q+1,\ q\in\mathbb{Z}$

In this case,

$$
\begin{aligned}
x &= 2^{n-1}(2^{n}-1)\\
&= 2^{4q}(2^{4q+1}-1)\\
&= 16^{q}(2\cdot 16^{q}-1)\\
&\equiv 6(2\cdot 6-1) \bmod 10.\\
&\equiv 66 \pmod{10} \equiv 6 \pmod{10}.
\end{aligned}
$$

So, in this case X ends in 6.

case 3; $n=4q+3,\ q\in\mathbb{Z}$

In this case,

$$
\begin{aligned}
x &= 2^{n-1}(2^{n}-1)\\
&= 2^{4q+2}(2^{4q+3}-1)\\
&= 4\cdot 16^{q}(8\cdot 16^{q}-1)
\end{aligned}
$$

$$\equiv 4 \cdot 6\,(8 \cdot 6 - 1) \pmod{10}.$$

$$\equiv (24)(47) \pmod{10}$$

$$\equiv (4)(7) \pmod{10}$$

$$\equiv 28 \pmod{10}$$

$$\equiv 8 \pmod{10}$$

So, x ends in an 8.

So, no matter what n is, x ends in 6 or 8. $\blacksquare$