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Weds

Week 6

So far we've characterized the even perfect numbers. They correspond to Mersenne primes.

There are 51 known even perfect numbers.

So far no one has found an odd perfect number. But if an odd perfect number exists it must satisfy a bunch of properties that various people have proven.

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Note:

Theorem 64 (Euler, 1707–1783)

Let n be an odd perfect number. Then the prime factorization of n is of the form

$$n = q^{e} p_1^{2a_1} p_2^{2a_2} \cdots p_r^{2a_r}$$

where $q, p_1, \ldots, p_r$ are odd primes and $q \equiv 1 \pmod 4$ and $e \equiv 1 \pmod 4$.

Note: The p_i's can be 1 or 3 mod 4.

proof:

Let n be an odd perfect number.

Suppose n's prime factorization is

$$n = l_1^{b_1} l_2^{b_2} \cdots l_s^{b_s}$$

where $l_1, l_2, \ldots, l_s$ are distinct primes and $b_1, b_2, \ldots, b_s$ are positive integers.

Since n is odd, all the l_i are odd primes.

Since n is perfect $\sigma(n) = 2n$. So,

$$\boxed{\sigma(x) = \sum_{d \mid x} d}$$

$$2n = \sigma(n) = \sigma(l_1^{b_1} l_2^{b_2} \cdots l_s^{b_s}) = \sigma(l_1^{b_1})\sigma(l_2^{b_2})\cdots\sigma(l_s^{b_s}).$$

Since n is odd and $2n = \sigma(l_1^{b_1})\sigma(l_2^{b_2})\cdots\sigma(l_s^{b_s})$

we know $2 \mid \sigma(l_1^{b_1})\sigma(l_2^{b_2})\cdots\sigma(l_s^{b_s})$

but $2^2 \nmid \sigma(l_1^{b_1})\sigma(l_2^{b_2})\cdots\sigma(l_s^{b_s})$.

Because $2 \mid 2n$, but $2^2 \nmid 2n$.

So exactly one of $\sigma(l_1^{b_1}), \sigma(l_2^{b_2}), \ldots, \sigma(l_s^{b_s})$ is even. And the even one isn't divisible by $4 = 2^2$.

Note that since l_i is prime we have

$$\sigma(l_i^{b_i}) = 1 + l_i + l_i^2 + \cdots + l_i^{b_i},$$

which is odd only when b_i is even.

Thus,

$$n = q^{e} p_1^{2a_1} p_2^{2a_2} \cdots p_r^{2a_r}$$

where $q, p_1, \ldots, p_r$ are primes and $e, a_1, \ldots, a_r$ are positive integers.

side work

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odd + odd = even

odd + odd + odd = odd

odd + odd + odd + odd = even.

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Here q is the prime
where $\sigma(q^e)$ is even but $2^2 \nmid \sigma(q^e)$
and $p_1, \dots, p_r$ are the l_i
that aren't even.

Here we mean $\sigma(l_i^{p_i})$ is not even

Now we look at
q and e.

We have

$$\sigma(q^e) = 1 + q + q^2 + \cdots + q^e$$

is even.

Since q is odd, this implies that e is odd.

Since q is odd, either $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$

Let's rule out the $q \equiv 3 \pmod 4$ case.

Suppose $q \equiv 3 \pmod 4$.

Then $q \equiv -1 \pmod 4$.

So,

$$\sigma(q^e) = 1 + q + q^2 + \cdots + q^e$$

$$\equiv 1 + (-1) + (-1)^2 + \cdots + (-1)^e \pmod 4$$

Since e is odd

$$\equiv 0 \pmod 4$$

But $\sigma(q^e) \equiv 0 \pmod 4$

means $4 \mid \sigma(q^e)$

which isn't the case.

So, $\boxed{q \equiv 1 \pmod 4.}$

Now we just need to show $e \equiv 1 \pmod 4$ and we are done.

Since e is odd either
$e \equiv 1 \pmod 4$ or $e \equiv 3 \pmod 4$.

Let's rule out the $e \equiv 3 \pmod 4$ case.

Suppoe $e \equiv 3 \pmod 4$.

Since $q \equiv 1 \pmod 4$, we have

$$\sigma(q^e) = 1 + q + q^2 + \cdots + q^e$$

$$\equiv 1 + 1 + 1^2 + \cdots + 1^e \pmod 4$$

$$\equiv (e+1) \pmod 4$$

$$\equiv (3+1) \pmod 4$$

$$\equiv 4 \pmod 4 \equiv 0 \pmod 4$$

So, $\sigma(q^e) \equiv 0 \pmod 4$.

Thus, $4 \mid \sigma(q^e)$

which isn't the case.

So, $e \equiv 1 \pmod 4$.

Thus, $n = q^e p_1^{2a_1} \cdots p_r^{2a_r}$

where $q, p_1, p_2, \ldots, p_r$ are primes and $q \equiv 1 \pmod 4$

and $e \equiv 1 \pmod 4$. $\boxed{/\!/}$

Note 65:

q is sometimes called the special/Euler prime of n.

Corollary 66:

Let n be an odd perfect number.

Then $n = q^e m^2$ where q is a prime, e and m are positive integers. And $q \equiv e \equiv 1 \pmod{4}$.

Moreover, this implies that $n \equiv 1 \pmod 4$

<u>proof:</u>

By Thm 64,

$$n = q^e p_1^{2a_1} \cdots p_r^{2a_r}$$

where $q, p_1, \ldots, p_r$ are odd primes.

and $e, a_1, \ldots, a_r$ are positive integers and $q \equiv e \equiv 1 \pmod 4$.

Let $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$.

Then, $n = q^e m^2$.

Since m is odd either $m \equiv 1 \pmod 4$ or $m \equiv 3 \pmod 4$.

So

or

So

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So, either

$$m^2 \equiv 1^2 \pmod 4 \equiv 1 \pmod 4$$

or

$$m^2 \equiv 3^2 \pmod 4 \equiv 9 \pmod 4 \equiv 1 \pmod 4.$$

So, in either case $m^2 \equiv 1 \pmod 4$.

Thus, since $q \equiv 1 \pmod 4$,

$$n = q^e m^2 \equiv 1^e \cdot 1 \pmod 4 \equiv 1 \pmod 4.$$

$\boxtimes$

Corollary 67:

There are no odd perfect numbers n with $n \equiv 3 \pmod 4$.