3/9
Monday
week 8

Def 71 Given a positive integer n define

$$I(n) = \frac{\sigma(n)}{n}$$

$I(n)$ is called the abundancy index/ratio of n.

Ex 72

$$I(4) = \frac{\sigma(4)}{4} = \frac{1+2+4}{4} = \frac{7}{4} < 2$$

$$I(6) = \frac{\sigma(6)}{6} = \frac{1+2+3+6}{6} = \frac{12}{6} = 2$$

Fact 73 Let n be a positive integer. Then, n is perfect iff $\sigma(n) = 2n$ iff $I(n) = \frac{\sigma(n)}{n} = 2$.

ger.

Fact 74

Let n be a positive integer where $n \geq 2$.
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n, where $p_1, p_2, \ldots, p_k$ are distinct primes.

Then

$$I(n) = I(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \frac{\sigma(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k})}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$$

$$\underset{\text{Thm 35}}{=} \frac{\sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})\cdots\sigma(p_k^{\alpha_k})}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}} = \frac{\sigma(p_1^{\alpha_1})}{p_1^{\alpha_1}} \cdot \frac{\sigma(p_2^{\alpha_2})}{p_2^{\alpha_2}} \cdots \frac{\sigma(p_k^{\alpha_k})}{p_k^{\alpha_k}}$$

$$= I(p_1^{\alpha_1}) I(p_2^{\alpha_2}) \cdots I(p_k^{\alpha_k})$$

Fact 75 Let p be prime and α be a positive integer.
Then

$$I(p^\alpha) = \frac{\sigma(p^\alpha)}{p^\alpha} = \frac{1+p+p^2+\cdots+p^\alpha}{p^\alpha}$$

$$= 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^\alpha}$$

Def 76

Let n be a positive integer.
Define $\omega(n)$ to be the number of distinct primes in the prime factorization of n.

Ex 77

$\omega(125) = \omega(5^3) = 1$

$\omega(12) = \omega(2^2 \cdot 3^1) = 2$

$\omega(7^3 \cdot 11^{105} \cdot 13^2 \cdot 17^1) = 4$

We are going to show that

$\omega(n) \geq 4$

if n is an odd perfect number.

Thm 78

Let n be an odd perfect number.

Then $\omega(n) \geq 2$.

Proof: Suppose n is an odd perfect number and $\omega(n) = 1$.

Then $n = p^\alpha$ where p is an odd prime and α is a positive integer.

Since n is perfect, $I(n) = 2$.

So,

$$
\begin{aligned}
2 &= I(n) \\
&= I(p^{\alpha}) \\
&= 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\alpha}} \\
&< \underbrace{1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots}_{\sum_{k=0}^{\infty} \left(\frac{1}{p}\right)^k}
\end{aligned}
$$

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Geometric series

$$1+r+r^2+r^3+\cdots = \frac{1}{1-r}$$

when $-1<r<1$

$$\frac{1}{1-\frac{1}{p}} = \frac{p}{p-1} \le \frac{3}{2} = 1.5.$$

$p \geq 3$ & lemma 79

This is a contradiction since $2 < 1.5$ is nonsense. Thus, $w(n) \geq 2$. $\blacksquare$

Lemma 79 If $p \geq c > 1$, then $\dfrac{p}{p-1} \le \dfrac{c}{c-1}$

pf of lemma:

$\boxed{p \geq c > 1} \Rightarrow \boxed{-p \le -c} \Rightarrow \boxed{pc - p \le pc - c}$

$\Rightarrow \boxed{p(c-1) \le c(p-1)} \Rightarrow \boxed{\dfrac{p}{p-1} \le \dfrac{c}{c-1}}$ $\blacksquare$

Thm 80

Let n be an odd perfect number.

Then $\omega(n) \geq 3$.

Proof: Suppose n is an odd perfect number. We know from Thm 78 that $\omega(n) \geq 2$. Let's rule out $\omega(n) = 2$.

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2}$

where p_1, p_2 are odd primes

and α_1, α_2 are positive integers.

We know $p_1 \geq 3$ and $p_2 \geq 5$.

Since n is perfect,

we know $I(n) = \frac{\sigma(n)}{n} = 2$

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$1 + r + r^2 + r^3 + \cdots$

$= \frac{1}{1-r}$

if $-1 < r$

ect

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2.

We get that

$$2 = I(n) = I\left(p_1^{\alpha_1} p_2^{\alpha_2}\right) \overset{\text{Fact 74}}{=} I\left(p_1^{\alpha_1}\right) I\left(p_2^{\alpha_2}\right)$$

$$\overset{\text{Fact 75}}{=} \left(1+\frac{1}{p_1}+\frac{1}{p_1^2}+\dots+\frac{1}{p_1^{\alpha_1}}\right)\left(1+\frac{1}{p_2}+\frac{1}{p_2^2}+\dots+\frac{1}{p_2^{\alpha_2}}\right)$$

$$< \left(1+\frac{1}{p_1}+\frac{1}{p_1^2}+\frac{1}{p_1^3}+\dots\right)\left(1+\frac{1}{p_2}+\frac{1}{p_2^2}+\frac{1}{p_2^3}+\dots\right)$$

$$= \left(\frac{1}{1-\frac{1}{p_1}}\right)\cdot\left(\frac{1}{1-\frac{1}{p_2}}\right) = \frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1} \le \frac{3}{2}\cdot\frac{5}{4} = \frac{15}{8} = 1.875$$

Lemma 79
$p_1 \geq 3$, $p_2 \geq 5$

geometric sum

$$1+r+r^2+r^3+\dots = \frac{1}{1-r} \quad \text{if } -1<r<1$$

Contradiction.
$2 < 1.875$??
Nonsense!!

$\blacksquare$

Theorem 81

Let n be an odd perfect number.
Then $\omega(n) \geq 4$.

<u>pf:</u> Let n be an odd perfect number. We already showed $\omega(n) \geq 3$.
Let's rule out $\omega(n) = 3$.

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$
is the prime factorization of n.
Since n is odd, the p_i are odd.
So we can assume
$p_1 \geq 3,\ p_2 \geq 5,\ p_3 \geq 7.$

We now show $3 \mid n$.
If $3 \nmid n$, then

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$\Rightarrow p_1 \geq 5,\ p_2 \geq 7,\ p_3 \geq 11.$

Then, by the same ideas as Thm 80 we would have

$$2 = I(n) = I(p_1^{\alpha_1}) I(p_2^{\alpha_2}) I(p_3^{\alpha_3})$$

$$< \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right)\left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots\right)\left(1 + \frac{1}{p_3} + \frac{1}{p_3^2} + \cdots\right)$$

$$= \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdot \frac{p_3}{p_3 - 1} \leq \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10}$$

$$= \frac{77}{48} \approx 1.6041$$

Contradiction.

$\boxed{\text{So we can assume } p_1 = 3}$

We can also show $p_2 = 5$.

If $p_1 = 3,\ p_2 \geq 7,\ p_3 \geq 11$

then as before

$$2 = I(n) < \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdot \frac{p_3}{p_3 - 1}$$

$$\leq \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10}$$

$$= \frac{231}{120} \approx 1.925$$

Contradiction

So, $\boxed{p_1 = 3 \text{ and } p_2 = 5 \text{ and } p_3 \geqslant 7}$

same as before

Thus,

$$2 = I(n) < \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdot \frac{p_3}{p_3 - 1}$$

$$\leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{p_3}{p_3 - 1}$$

So, $\frac{16}{15} < \frac{p_3}{p_3 - 1}$. So, $16p_3 - 16 < 15p_3$.

Thus, $p_3 < 16$.

Summary: $n = 3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3}$
where $p_3 = 7, 11$, or 13.

Euler: If n is an odd perfect number then $n = q^e p_1^{2e_1} p_2^{2e_2} \cdots p_k^{2e_k}$
$q \equiv e \equiv 1 \pmod 4$. So, e is odd.

Thm 64

case 1: $n = 3^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3}$

Since $3 \equiv 7 \equiv 3 \pmod 4$
and $5 \equiv 1 \pmod 4$,
by Euler, $\alpha_1 \geq 2$ and $\alpha_3 \geq 2$.

So,

$$2 = I(n)$$
$$= \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{\alpha_1}}\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^{\alpha_2}}\right)\left(1 + \frac{1}{7} + \frac{1}{7^2} + \cdots + \frac{1}{7^{\alpha_3}}\right)$$
$$\geq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7} + \frac{1}{7^2}\right) = \frac{494}{245} \approx 2.0163\ldots$$

Contradiction.

Case 2: $n = 3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3}$

Since $3 \equiv 11 \equiv 3 \pmod 4$ and $5 \equiv 1 \pmod 4$

we get by Euler, $\alpha_2 \geq 1$, $\alpha_1 \geq 2$, $\alpha_3 \geq 2$.

Case 2A: $\alpha_2 = 1$

Then,

$$2 = I(n) < \left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{11}+\frac{1}{11^2}+\cdots\right)$$

$$< \frac{3}{2}\cdot\frac{6}{5}\cdot\frac{11}{10} = \frac{99}{50}$$ contradiction.

Case 2B: $\alpha_2 \geq 2$

If $\alpha_1 = 2$, then

$$2n = \sigma(n) = \sigma(3^2)\sigma(5^{\alpha_2})\sigma(11^{\alpha_3})$$
$$= (1+3+3^2)\sigma(5^{\alpha_2})\sigma(11^{\alpha_3})$$
$$= 13\,\sigma(5^{\alpha_2})\sigma(11^{\alpha_3})$$

But $13 \nmid 2n$.

So this case can't happen.

(11^{α_3})

$)\sigma(11^{\alpha_3})$

$11^{\alpha_3})$

en.

So, $\alpha_1 \geq 4$, $\alpha_2 \geq 2$, $\alpha_3 \geq 2$.

Then

$$
\begin{aligned}
2 = I(n) &= \left(1+\frac{1}{3}+\cdots+\frac{1}{3^{\alpha_1}}\right)\left(1+\frac{1}{5}+\cdots+\frac{1}{5^{\alpha_2}}\right)\left(1+\frac{1}{11}+\cdots+\frac{1}{11^{\alpha_3}}\right) \\
&\geq \left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}\right)\left(1+\frac{1}{5}+\frac{1}{5^2}\right)\left(1+\frac{1}{11}+\frac{1}{11^2}\right) = \frac{4123}{2025} \approx 2.036.
\end{aligned}
$$

Contradiction.

Case 3: $n = 3^{\alpha_1} 5^{\alpha_2} 13^{\alpha_3}$

Case 3A: $\alpha_2 = 1$

Then

$$2 = I(n) = \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{\alpha_1}}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{13} + \cdots + \frac{1}{13^{\alpha_3}}\right)$$

$$< \frac{3}{2} \cdot \left(1 + \frac{1}{5}\right) \cdot \frac{13}{12} = \frac{39}{20}$$

Contradiction

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Case 3B: $\alpha_2 \geq 2$

Since $3 \not\equiv 1 \pmod 4$, Euler says that α_1 is even.

Case 3B-1: $\alpha_1 = 2$

$$2 = I(n) = \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5} + \cdots + \frac{1}{5^{\alpha_2}}\right)\left(1 + \frac{1}{13} + \cdots + \frac{1}{13^{\alpha_3}}\right)$$

$$< \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \cdot \frac{5}{4} \cdot \frac{13}{12} = \frac{845}{432} \approx 1.956$$

Can't happen

Case 3B-2 : $\alpha_1 \geq 4$

If $\alpha_3 = 1$, then $2n = \sigma(n) = \sigma(3^{\alpha_1})\sigma(5^{\alpha_2})\sigma(13)$

$$= 14\,\sigma(3^{\alpha_1})\sigma(5^{\alpha_2})$$

Then $14 \mid 2n$. Or $7 \mid n$. Cant happen.

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So, $\alpha_3 \geqslant 2$.

Thus,

$$2 = I(n) \geqslant \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right)\left(1 + \frac{1}{5} + \frac{1}{5^2}\right)\left(1 + \frac{1}{13} + \frac{1}{13^2}\right)$$

$$= \frac{228811}{114075} \approx 2.00579\ldots$$

Contradiction.

$\blacksquare$

If n is an odd perfect number, then

1. n is divisible by a prime p with $p > 10^8$ [Goto & Ohno, 2008]

2. $n > 10^{1500}$ [Ochem & Rao, 2012]

3. $105 \nmid n$, $[105 = 3 \cdot 5 \cdot 7]$

4. $n \equiv 1 \pmod{12}$ or $n \equiv 117 \pmod{468}$ or $n \equiv 81 \pmod{324}$

5. $n = q^{\alpha} p_1^{2e_1} p_2^{2e_2} \cdots p_k^{2e_k}$

   where $q, p_1, p_2, \ldots, p_k$ are distinct odd primes, $\alpha, e_1, \ldots, e_k$ are positive integers and $\alpha \equiv q \equiv 1 \pmod{4}$ [Euler]

6. n has at least 10 distinct prime factors [Nielsen, 2015]