2/19
Weds
Week 5

8.4 continued...

Recall:

Divergence test:

If $\lim_{k\to\infty} a_k \neq 0$,

then $\sum_{k=1}^{\infty} a_k$ diverges

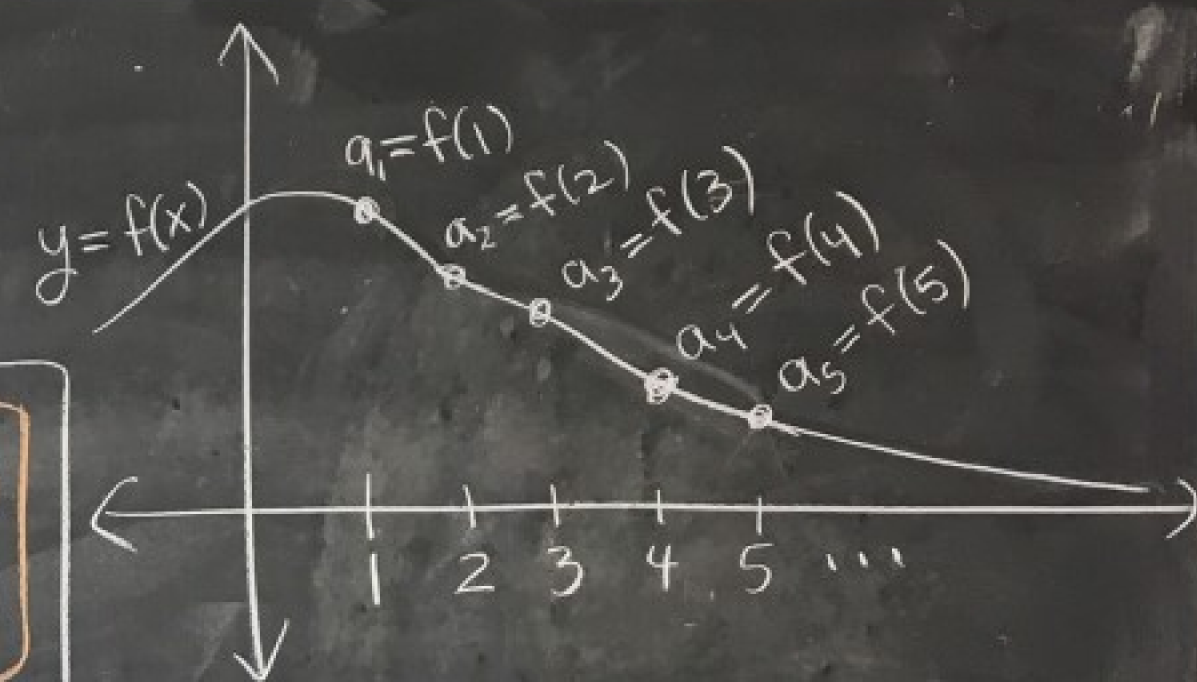
Integral test

Suppose that

① $f(x)$ is continuous for $x \geq 1$

② $f(x)$ is positive for $x \geq 1$

③ $f(x)$ is decreasing for $x \geq 1$ ← $f'(x) < 0$ for $x \geq 1$



Let $a_k = f(k)$.

- If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
- If $\int_1^\infty f(x)\,dx$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

Note: You don't have to start at $k=1$. You can start at any $k=5$ then you just check conditions ①,②,③ for $x \geq 5$.

Note: If $\int_1^\infty f(x)\,dx$ converges, then

$$\int_1^\infty f(x)\,dx \neq \sum_{k=1}^\infty a_k$$

sum of blue boxes is $\sum_{k=1}^\infty a_k$. Orange region is $\int_1^\infty f(x)\,dx$

Ex: Consider

$$\sum_{k=1}^\infty \frac{\ln(k)}{k}$$

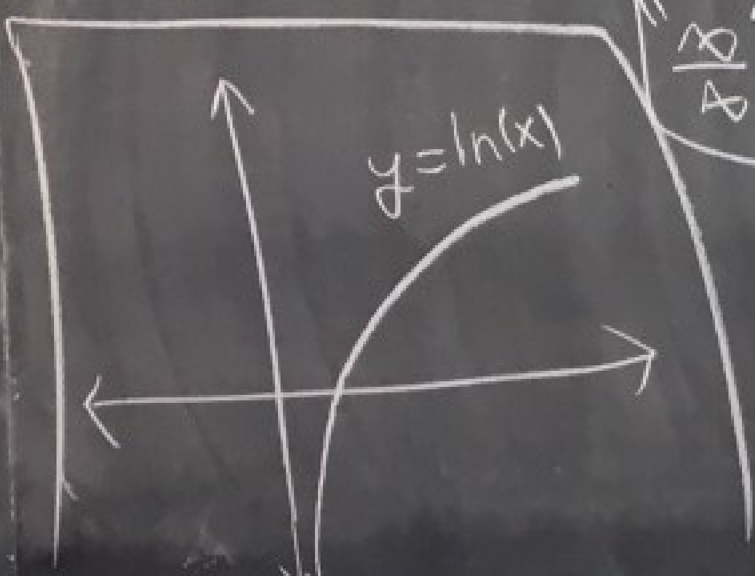
$$= \frac{\ln(1)}{1} + \frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \dots$$

Does this converge or diverge?

step 1: Does $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ pass

the divergence test ?

$$\lim_{k\to\infty} \frac{\ln(k)}{k} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{L'H}{=}} \lim_{k\to\infty} \frac{1/k}{1} = \lim_{k\to\infty} \frac{1}{k} = 0$$



So, the divergence test does not apply.

Step 2: Use some other test.

We will use the integral test.

Check the three conditions of the integral test.

$$\boxed{\text{Let } f(x) = \frac{\ln(x)}{x}}$$

Note that $\frac{\ln(1)}{1} = 0$,

So our sum is $\sum_{k=2}^{\infty} \frac{\ln(k)}{k}$

① $f(x) = \frac{\ln(x)}{x}$ is continuous for $x \geq 2$.

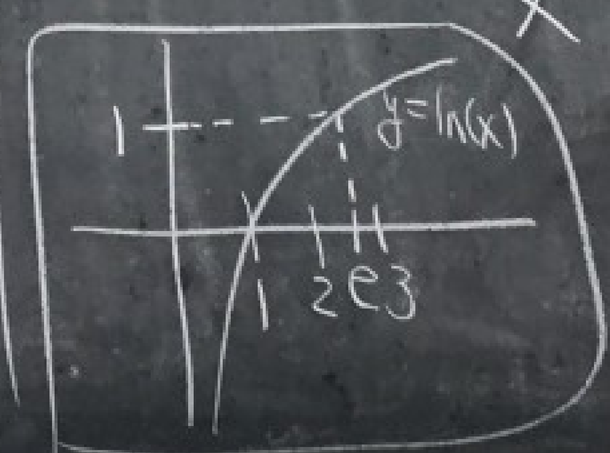
② $f(x) = \frac{\ln(x)}{x} > 0$ when $x \geq 2$

($\ln(x) > 0$ when $x > 1$)

③ Is $f(x)$ decreasing when $x \geq 2$?

$$f'(x) = \frac{\left(\frac{1}{x}\right)\cdot x - (1)\ln(x)}{x^2}.$$

$$f'(x) = \frac{1 - \ln(x)}{x^2} < 0$$



If $x \geq e \approx 2.7$, then $\ln(x) > 1$ So, $0 > 1 - \ln(x)$	If $x \geq e$, then $1 - \ln(x) < 0$ $x^2 > 0$

So, $f(x)$ is decreasing for $x \geq 3$.

So we can use the integral test.

$$\int_2^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\int_2^t \frac{\ln(x)}{x}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}(\ln(x))^2\right]_2^t = \lim_{t\to\infty} \frac{1}{2}\underbrace{[\ln(t)]^2}_{\to\infty} - \frac{1}{2}[\ln(2)]^2$$

$$= \infty$$

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{u^2}{2}+C$$

$$u=\ln(x)$$
$$du=\frac{1}{x}\,dx$$

$$= \frac{1}{2}[\ln(x)]^2+C$$

Since $\int_2^\infty \frac{\ln(x)}{x}\,dx$ diverges, $\sum_{k=2}^{\infty} \frac{\ln(x)}{x}$ diverges