2/20
Thursday
Week 5

8.4 continued...

Ex: Does $\sum_{k=0}^{\infty} \frac{1}{k^2+4} = \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \cdots$

converge or diverge?

S

L

L

Does it pass the divergence test?

$$\lim_{k\to\infty} \frac{1}{k^2+4} = 0$$

So the divergence test doesn't apply.

Let's try to use the integral test.

Let $f(x) = \dfrac{1}{x^2+4}$.

$$(x^2+4)^2 \geq 0$$
$$-2x < 0 \text{ when } x>0$$

① $f(x)=\frac{1}{x^2+4} > 0$ for all x.

② $f(x)=\frac{1}{x^2+4}$ is continuous

③ $f'(x)=\left[(x^2+4)^{-1}\right]' = -(x^2+4)^{-2}\cdot(2x) = \frac{-2x}{(x^2+4)^2} < 0$ when $x>0$.

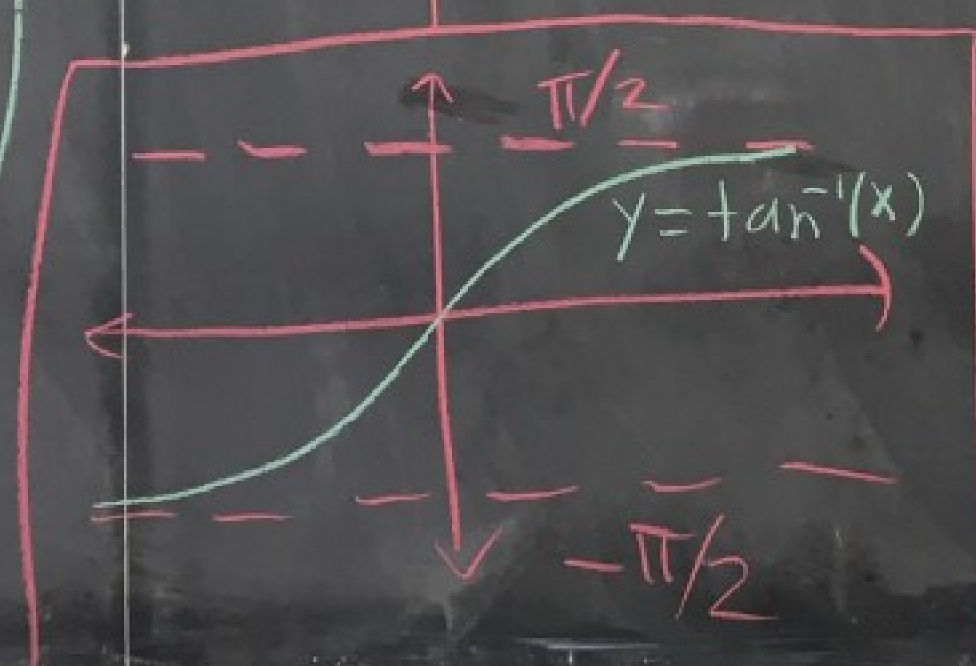
Now we calculate $\int_0^\infty \frac{dx}{x^2+4} = \frac{1}{4} \lim_{t\to\infty}\int_0^t \frac{dx}{\frac{x^2}{4}+1} = \frac{1}{4}\lim_{t\to\infty}\int_0^t\frac{dx}{(\frac{x}{2})^2+1}$

$$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

$$= \frac{1}{4}\left[\lim_{t\to\infty} 2\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^t\right] = \frac{1}{2}\lim_{t\to\infty}\Big[\underbrace{\tan^{-1}\left(\frac{t}{2}\right)}_{\to\ \pi/2} - \underbrace{\tan^{-1}(0)}_{0}\Big]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{2}\right) = \frac{1}{\left(\frac{x}{2}\right)^2+1}\cdot\frac{1}{2}$$



Since $\int_0^\infty \frac{dx}{x^2+4}$ converges

by the integral test

$\sum_{k=0}^{\infty} \frac{1}{k^2+4}$ converges.

Ex: (p-series)

Does $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge or diverge? How does this depend on p?

case: $p=1$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

is the harmonic series, which diverges.

case: $p<0$

In this case,

$$\lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} k^{-p} = \infty$$

$$\boxed{-p>0}$$

Since $\lim_{k\to\infty} \frac{1}{k^p} \neq 0$, by the divergence test, $\sum_{k=0}^{\infty} \frac{1}{k^p}$ diverges

$$\boxed{\text{Ex: } p=-2 \quad \frac{1}{k^p} = \frac{1}{k^{-2}} = k^2}$$

case: $p=0$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1+1+1+1+\dots$$

$$\boxed{\frac{1}{k^0}=1}$$

diverges

case: $p>0,\ p\neq 1$

Integral test time

$f(x)=\frac{1}{x^p}=x^{-p}$

① $f(x)=\frac{1}{x^p}>0$ if $x>0$.

② $f(x)=\frac{1}{x^p}$ is continuous for $x>0$

③ $f'(x)=-px^{-p-1}=-p\left(\frac{1}{x^{p+1}}\right)<0$ if $x>0$

$$\boxed{\begin{array}{l} -p<0 \\ \frac{1}{x^{p+1}}>0 \text{ when } x>0 \end{array}}$$

So, f is decreasing when $x>0$.

$\int_1^\infty \frac{1}{x^p}\,dx$

$=\left(\frac{1}{-p+1}\right)\lim_{t\to\infty}$

If $0<p<1$,

If $1<p$,

is decreasing

$x > 0$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^t$$

$$= \left(\frac{1}{-p+1}\right) \lim_{t\to\infty} \left[t^{-p+1} - 1^{-p+1} \right] = \left(\frac{1}{-p+1}\right) \lim_{t\to\infty} \left[t^{1-p} \right] \quad (*)$$

If $0<p<1$, then $1-p>0$ and the limit in $(*)$ is infinite.

If $1<p$, then $1-p<0$ and the limit in $(*)$ is 0.

In this case $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p>1$ and diverges if $0<p<1$.

>0

Summary

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$ converges if $p > 1$

and diverges otherwise