Mon
2/24
Week 6

8.4 continued...

Recall from last time

p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p > 1$

otherwise it diverges

Ex:

$$\sum_{k=1}^{\infty} k^{-5} = \sum_{k=1}^{\infty} \frac{1}{k^5} \quad \left(p = 5 > 1\right) \quad \text{So, } \sum_{k=1}^{\infty} \frac{1}{k^5} \text{ converges.}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/4}} \quad \left(p = \frac{3}{4} < 1\right). \quad \text{So, } \sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}} \text{ diverges.}$$

Properties of convergent series

Suppose $\sum a_k$ converges to A

and $\sum b_k$ converges to B.

① If c is a constant, $\sum c a_k = c \sum a_k = cA$.

② $\sum (a_k + b_k) = \sum a_k + \sum b_k = A + B$

③ $\sum (a_k - b_k) = \sum a_k - \sum b_k = A - B$

Ex: Consider $\displaystyle\sum_{k=1}^{\infty}\left[5\left(\frac{2}{3}\right)^k - \frac{2^{k-1}}{7^k}\right]$

$$\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^k = 5\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = 5\left[\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots\right]$$

$$= 5\cdot\frac{2}{3}\left[1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots\right] = 5\cdot\frac{2}{3}\cdot\left[\frac{1}{1-\frac{2}{3}}\right] = 5\cdot\frac{2}{3}\cdot 3 = \boxed{10}$$

$$\boxed{1+r+r^2+r^3+\cdots = \frac{1}{1-r} \quad \text{if } -1<r<1}$$

Method 2

$$5\left[\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots\right]$$

$$=5\left[\underbrace{-1+1}_{0}+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots\right]$$

$$=5\left[-1+\frac{1}{1-\frac{2}{3}}\right]=5\left[-1+3\right]=10$$

$$\sum_{k=1}^{\infty}\frac{2^{k-1}}{7^k}=\frac{1}{7}+\frac{2}{7^2}+\frac{2^2}{7^3}+\frac{2^3}{7^4}+\cdots$$

$$=\frac{1}{7}\left[1+\frac{2}{7}+\frac{2^2}{7^2}+\frac{2^3}{7^3}+\cdots\right]$$

$$=\frac{1}{7}\left[\frac{1}{1-\frac{2}{7}}\right]=\frac{1}{7}\left[\frac{1}{\left(\frac{5}{7}\right)}\right]$$

$$\boxed{\begin{array}{c} r=\frac{2}{7} \\ -1<r<1 \end{array}}$$

$$=\frac{1}{7}\cdot\frac{7}{5}=\frac{1}{5}$$

So, $$\sum_{k=1}^{\infty}\left[5\left(\frac{2}{3}\right)^k - \frac{2^{k-1}}{7^k}\right] = \sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^k - \sum_{k=1}^{\infty} \frac{2^{k-1}}{7^k}$$

$$= 10 - \frac{1}{5} = \boxed{\frac{49}{5}}$$

8.5 — Ratio, Root, Comparison Tests

Ratio test: Let $\sum a_k$ be an infinite series where each $a_k > 0$

Let $r = \lim_{k\to\infty} \left(\frac{a_{k+1}}{a_k}\right)$

① If $0 \le r < 1$, then the series converges.

② If $r > 1$ (this includes "$r = \infty$"), then the series diverges.

③ If $r = 1$, the test is inconclusive.

Ex: Consider

$$\sum_{k=1}^{\infty} \frac{23^k}{k!} = \frac{23}{1!} + \frac{23^2}{2!} + \frac{23^3}{3!} + \cdots$$

Recall factorial

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 120$

Use ratio test. All terms are positive ✓

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\left(\frac{23^{k+1}}{(k+1)!}\right)}{\left(\frac{23^k}{k!}\right)} =$$

$$= \lim_{k\to\infty} \frac{23^{k+1}}{(k+1)!} \cdot \frac{k!}{23^k} = \lim_{k\to\infty} 23 \cdot \frac{k!}{(k+1)!} = \lim_{k\to\infty} 23 \cdot \frac{k!}{(k+1)\cdot[k!]}$$

$$\boxed{23^{k+1} = 23 \cdot 23^k}$$

$$= \lim_{k\to\infty} 23 \cdot \frac{1}{k+1}$$

$$= \lim_{k\to\infty} \frac{23}{k+1} = 0 = r$$

Since $0 \le r < 1$, the series $\sum_{k=1}^{\infty} \frac{23^k}{k!}$ converges.

Factorial

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$\quad = 6 \cdot [5!]$

$(n+1)! = (n+1) \cdot [n!]$

For fun (We will learn this later)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for any } x$$

$$e^{23} = \sum_{k=0}^{\infty} \frac{23^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{23^k}{k!}$$

$$\boxed{0! = 1}$$

So, $\displaystyle\sum_{k=1}^{\infty} \frac{23^k}{k!} = e^{23} - 1.$