Math 2120

3/23/20

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Recall that the Taylor series for $f(x)$ centered at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Ex: Find the Taylor series centered at $a=0$ (or Maclaurin series) for $f(x)=\sin(x)$.

$$\left.\begin{aligned} f(x) &= \sin(x)\\ f'(x) &= \cos(x)\\ f''(x) &= -\sin(x)\\ f'''(x) &= -\cos(x) \end{aligned}\right\} \text{this part repeats}$$

$$\begin{aligned} f^{(4)}(x) &= \sin(x)\\ f^{(5)}(x) &= \cos(x)\\ f^{(6)}(x) &= -\sin(x)\\ f^{(7)}(x) &= -\cos(x)\\ &\vdots \end{aligned}$$

plug in $a=0$

$$\left.\begin{aligned} f(0) &= \sin(0) = 0\\ f^{(1)}(0) &= \cos(0) = 1\\ f^{(2)}(0) &= -\sin(0) = 0\\ f^{(3)}(0) &= -\cos(0) = -1 \end{aligned}\right\} \text{this part repeats}$$

$$\left.\begin{aligned} f^{(4)}(0) &= \sin(0) = 0\\ f^{(5)}(0) &= \cos(0) = 1\\ f^{(6)}(0) &= -\sin(0) = 0\\ f^{(7)}(0) &= -\cos(0) = -1 \end{aligned}\right\} \text{repeats}$$

$$\vdots$$

Maclaurin series / Taylor series centered at $a=0$

We calculated these on the last page

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots$$

$$= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

Answer

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

Check:

$k=0$: $(-1)^0 \frac{x^{2(0)+1}}{(2(0)+1)!} = 1 \cdot \frac{x^1}{1!} = \frac{1}{1!}x$

$k=1$:

$(-1)^1 \frac{x^{2(1)+1}}{(2(1)+1)!} = -\frac{x^3}{3!}$

odd numbers

$2k+1$

$2k-1$

So far we've got this:

function	Maclaurin series / Taylor series at $a=0$
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\sin(x)$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Question: If you compute the Taylor series for a function $f(x)$, does $f(x)$ actually equal $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$?

Answer: There are some technical theorems that allow you to prove when $f(x)$ equal its Taylor series.

One can show that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x.$$

Same for sin(x), That is,

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \text{for all } x.$$

How to get cos(x)? Take the derivative of sin(x).

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{for all } x.$$

Taking the derivative we get

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2k+1) x^{2k}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(2k+1)x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)[(2k)!]}(2k+1)x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

So,

$$\boxed{\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}} \quad \underbrace{\text{for all } x.}_{\text{since } \sin(x) \text{ converged for all } x \text{ also.}}$$

$$= \frac{(-1)^0}{0!}x^0 + \frac{(-1)^1}{2!}x^2 + \frac{(-1)^2}{4!}x^4 + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Recall,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

Ex: Calculate the following limit if it exists.

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

you could use L'Hospital's rule since it's a "$\frac{0}{0}$" situation

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - x}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right)$$

$$= \frac{1}{2} + \underbrace{\frac{0}{3!} + \frac{0^2}{4!} + \cdots}_{0} = \frac{1}{2}$$

Some Taylor series that we know. (All are Maclaurin series since $a=0$)

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x|<1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for } \underbrace{-1 \le x < 1}_{\text{in class we didn't check } x=-1}$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad \text{for } \underbrace{-1 \le x \le 1}_{\text{in class we didn't check } x=-1 \text{ and } x=1}$$

Question in class

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1-(1-x)) = -\sum_{k=1}^{\infty} \frac{(1-x)^k}{k}$$

$$\ln(x) = -\sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{k}$$

Workshop

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20 $$\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$$

Divergence test:

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+4}} = \lim_{k\to\infty} \frac{(1/k)\, k}{(1/k)\sqrt{k^2+4}}$$

$\sqrt{k^2+4} \approx \sqrt{k^2} \approx k$
when k is large

$$= \lim_{k\to\infty} \frac{1}{\sqrt{\frac{1}{k^2}}\sqrt{k^2+4}}$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{1+4/k^2}}$$

$$= \frac{1}{\sqrt{1+0}} = 1 \neq 0$$

So the series diverges!

Same problem

$$\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$$

$$f(x) = \frac{x}{\sqrt{x^2+4}} = \frac{x}{(x^2+4)^{1/2}}$$

① f is continuous ✓

② $f(x) > 0$ if $x > 0$

③ $$f'(x) = \frac{(1)\sqrt{x^2+4} - \left(\frac{1}{2}(x^2+4)^{-1/2} \cdot 2x\right) \cdot x}{\left((x^2+4)^{1/2}\right)^2}$$

$$= \frac{\sqrt{x^2+4} - \frac{x^2}{\sqrt{x^2+4}}}{x^2+4} \cdot \frac{\sqrt{x^2+4}}{\sqrt{x^2+4}}$$

$$= \frac{(x^2+4) - x^2}{(x^2+4)\sqrt{x^2+4}} = \frac{4}{(x^2+4)\sqrt{x^2+4}} > 0$$

So, ③ isn't satisfied. So, integral test can't be used. positive.

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Does $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ converge?

No. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$

$p = \frac{1}{3}$ series.

$p \not> 1$ so it diverges.

8.4 #47 Does this series converge and if so, what is the sum?

$$\sum_{k=1}^{\infty} \left[\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k \right]$$

$$= \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{5}{6}\right)^k + \sum_{k=1}^{\infty} \frac{3}{5}\left(\frac{7}{9}\right)^k$$

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$$\sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{5}{6}\right)^k = \frac{1}{3}\left[\left(\frac{5}{6}\right)^1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \ldots\right]$$

$$= \frac{1}{3}\cdot\frac{5}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \ldots\right]$$

$$= \frac{1}{3}\cdot\frac{5}{6}\left[\frac{1}{1-\frac{5}{6}}\right] = \frac{5}{3}$$

$$\sum_{k=1}^{\infty} \frac{3}{5}\cdot\left(\frac{7}{9}\right)^k = \frac{3}{5}\cdot\frac{7}{9}\left[1 + \frac{7}{9} + \left(\frac{7}{9}\right)^2 + \ldots\right]$$

$$= \frac{3}{5}\,\frac{7}{9}\left[\frac{1}{1 - 7/9}\right]$$

$$= \frac{21}{10}$$

So answer is $\frac{5}{3} + \frac{21}{10}$

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots$$

$$= \sum_{k=0}^{\infty} r^k$$

$$-1 < r < 1$$